Combinatorial Co-Workout

Exercises with Lukas and Bernd

Sunday, March 20, 2022

1. Fix five generic points in the affine plane $\mathbb{R}^2$, and draw the ten lines spanned by any pair of points. How many bounded regions does this line arrangement have? What is the characteristic polynomial? Now increase from five points to six or seven points.

2. Determine the rank 3 matroid M represented over the field $\mathbb{Q}(x,y)$ by the matrix
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & y & 0 & 1 & y \\ 0 & 1 & x & 0 & 1 & y & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1/(1-x) & x/(y(x-1)) & 1 \end{bmatrix}.$$

3. Let $\mathcal{R}(M)$ be the space of complex realizations of M, by nine points in $\mathbb{P}^2$, up to projective transformations. Give a description of $\mathcal{R}(M)$ by polynomial inequations.

4. Fix positive integers $u_1, u_2, \ldots, u_7$ and consider the polynomial function
$$(x,y) \;\mapsto\; x^{u_1} y^{u_2} (1-x)^{u_3} (1-y)^{u_4} (1-x-y)^{u_5} (1-xy)^{u_6} (xy-x-y)^{u_7}.$$
Determine the number of complex critical points (x,y), where the critical value is not 0.

5. Can you find $u_1, u_2, \ldots, u_7 \in \mathbb{N}$ such that all complex critical points are actually real?

6. Compute the Euler characteristic for the realization space $\mathcal{R}(M)$ of the matroid M.

7. Determine the tropical linear space (Bergman fan) of the matroid M. Draw a picture.

8. Describe the tropicalization of the very affine surface $\mathcal{R}(M)$. Draw a picture.

9. How about eight points in $\mathbb{P}^3$? With a view towards Exercises 2,3,4,5, what can you say about the realization space of the uniform rank 4 matroid on 8 elements? Dimension?

10. In your opinion, what is a **positive geometry**?

11. Is there any advantage of using OSCAR over using SAGEMATH? What is `MathRepo`?