Real-rooted and Hyperbolic Polynomials Tutorial Problem Set

Mario Kummer and Liam Solus

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Real-rooted polynomials:

1. Let $p \in \mathbb{R}[x]$ be a degree d polynomial with only nonnegative coefficients that is symmetric with respect to $n \geq d$ and real-rooted. Show that p is γ-positive. Show that the converse need not hold.

2. Let $p, q \in \mathbb{R}[x]$ real rooted with positive leading coefficients. Prove that the *Wronskian* $W(p, q) = p'q - q'p$ is nonpositive on $\mathbb{R}$ if and only if $p \preceq q$. In particular, we have that $(p')^2 - p \cdot q''$ is nonnegative for every real rooted polynomial p.

3. Use the Hermite-Biehler Theorem to prove the following:

   (a) $p \preceq \alpha p$ for all $\alpha \in \mathbb{R}$.

   (b) If $p \preceq q$ then $q \preceq -p$.

   (c) If $p \preceq q$ then $\alpha p \preceq \alpha q$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

4. If $p \preceq q$ and $p \not\equiv 0$ we say p is a *proper interleaver* of q. We say that the polynomials $(p_i)_{i=1}^n$ are 2-*compatible* if for all $i, j \in [n]$ the polynomial $\lambda_i p_i + \lambda_j p_j$ is real-rooted for all $\lambda_i, \lambda_j \geq 0$. A result of Chudnovsky and Seymour states that a sequence of polynomials $(p_i)_{i=1}^n$ is 2-compatible if and only if $p_1, \ldots, p_n$ have a common proper interleaver. Use this observation to prove that if $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ be two interlacing sequences then
$$p_1 q_n + p_2 q_{n-1} + \cdots + p_n q_1$$
is real-rooted.

5. The *independence polynomial* of a graph $G = (V, E)$ is $I(G; x) = \sum_{i \geq 0} \alpha_i x^i$ where α_i is the number of *independent sets* of size i in G; i.e., the sets of i vertices of G in which no two elements are adjacent in G. Show that the independence polynomial of the path and cycle on n vertices are real-rooted.

6. Let E_n^s be the s-Eulerian polynomial for $s = (s_1, \ldots, s_n)$ and let $s' = (s_1, \ldots, s_{n-1})$.

(a) Show that for all $i \in [s_n - 1]$

$$E_{s,i}(x) = \sum_{j=0}^{t_i - 1} x E_{s',j}(x) + \sum_{j=t_i}^{s_{n-1}-1} E_{s',j}(x),$$

where $t_i = \lceil i s_{n-1}/s_n \rceil$.

(b) Show that the above recursion maps interlacing sequences to interlacing sequences.

7. Let $(p_i)_{i=0}^n$ be a sequence of degree d real-rooted polynomials such that $p_{i-1} \preceq p_i$ for all $i \in [n]$ and $p_0 \preceq p_n$. Show that $(p_i)_{i=0}^n$ is interlacing.

8. Let $p, q, h \in \mathbb{R}[x]$ be degree d real-rooted polynomials with positive leading coefficients. Show that if $p \preceq q$ and $p \preceq h$ then for all $\lambda, \mu \geq 0$, $p \preceq \lambda q + \mu h$.

9. Show that $(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n)$.

10. Let Δ be a Boolean cell complex. Show that $f_{\mathrm{sd}(\Delta)} = \mathcal{E}(f_\Delta)$.

11. Show that a polynomial $p = \sum_{i=0}^d p_i x^i$ with only nonnegative coefficients is alternatingly increasing if and only if

$$0 \leq p_0 \leq p_d \leq p_1 \leq p_{d-1} \leq \cdots \leq p_{\lfloor \frac{d+1}{2} \rfloor}.$$

Multivariate stable polynomials:

1. Write down the symbols for the operations from Lemma 11 and prove their stability.

2. Prove that the elementary symmetric polynomials are strongly Rayleigh by using Newton's inequalities.

3. Let X the diagonal matrix with diagonal entries $x_1, \ldots, x_n$ and let A be a real symmetric $n \times n$ matrix. Let M be a matroid of rank r with the half-plane property. Prove that the sum of all principal $r \times r$ minors of $X + A$, that correspond to a basis of M, is a stable polynomial in $x_1, \ldots, x_n$.

4. Show that the elementary symmetric polynomial $e_{d,n}$ has a determinantal representation if and only if $d \leq 1$ or $n - d \leq 1$.

5. Prove that the class of matroids with the half-plane property is closed under taking minors and duals.

Lorentzian polynomials:

1. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ homogeneous of degree 2 with nonnegative coefficients. Show that h is Lorentzian if and only if h is stable.

2. Let $h = \sum_{i=0}^d a_i x_1^i x_2^{d-i}$. Prove that h is Lorentzian if and only if the sequence $a_1, \ldots, a_d$ is an an ultra log-concave sequence of nonnegative numbers with no internal zeros.

3. Construct a Lorentzian polynomial which is not stable.

4. Show that the elementary symmetric polynomial $e_{d,n}$ can be written as $\mathrm{vol}(x_1K_1 + \ldots + x_nK_n)$ for some convex bodies K_i if and only if $d \leq 1$ or $n - d \leq 1$.