I. Some old graph theory, with a logic perspective.

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Graphs without induced cycles of length ≥ 4.

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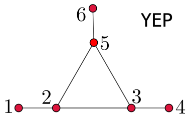

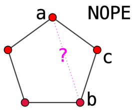

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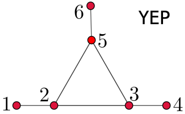

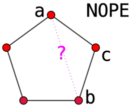

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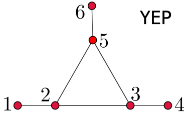

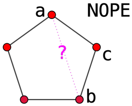

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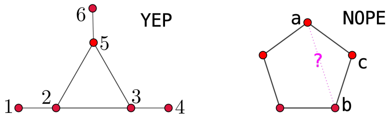

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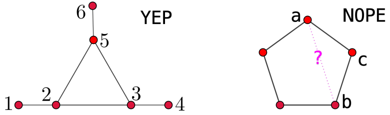

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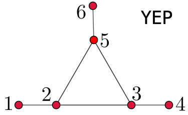

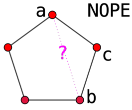

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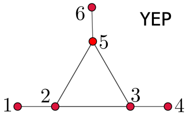

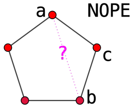

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"$\Rightarrow$": By Dirac's theorem, any graph without induced cycles of length ≥ 4 has a **simplicial** vertex, i.e. a vertex such that any two of its neighbors are connected by an edge.

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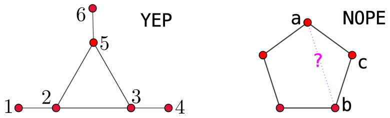

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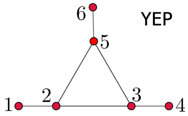

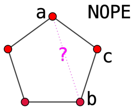

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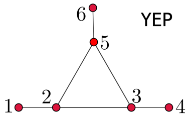

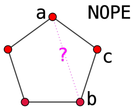

Via Dirac's theorem, chordal graphs can be characterized as the graphs that are either complete, or can be obtained recursively by joining two smaller chordal graphs whose intersection is complete.

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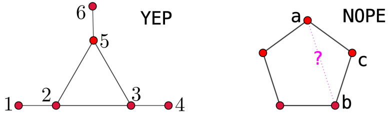

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It follows that chordal graphs can also be characterized as intersection graphs of some subtrees of some tree.

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Intersection graphs of curves between two horizontal lines.

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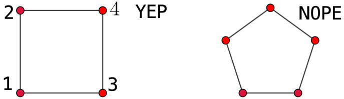

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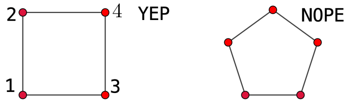

"$\Rightarrow$": Label curves left-to-right as they touch the bottom line. If curve 1 intersects curve 3, one of them must go across curve 2.
"$\Leftarrow$: By induction, $G - \{n\}$ is intersection graph of $n - 1$ curves between horizontal lines. Draw the last curve, and cross exactly those curves i such that $[i, n]$ is an edge of G. $\square$

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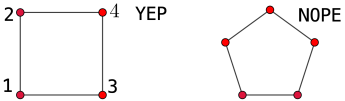

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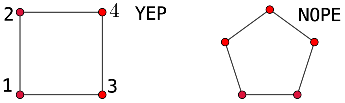

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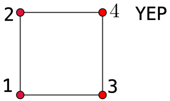

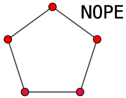

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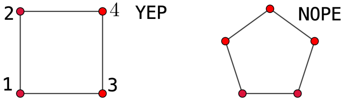

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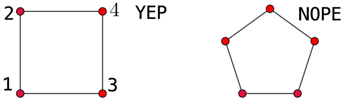

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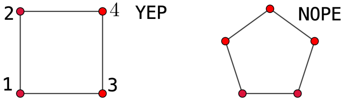

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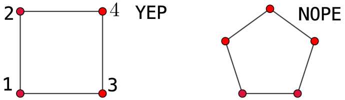

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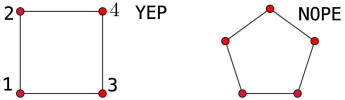

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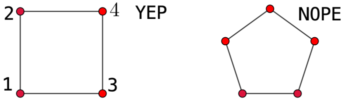

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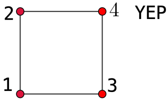

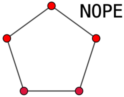

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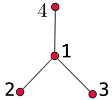

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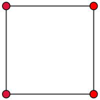

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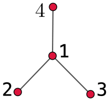

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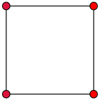

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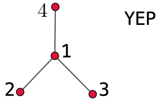
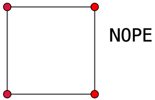
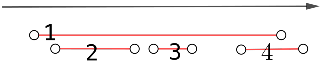




'$\Rightarrow$'. Swipe the real line left-to-right, label intervals as you encounter them (= order them by leftmost endpoint).



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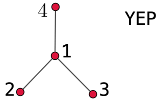

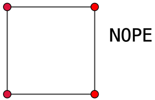

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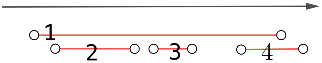

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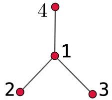

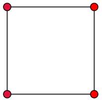

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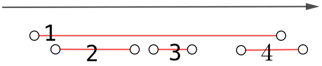

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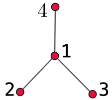

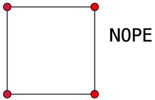

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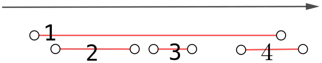

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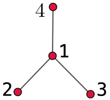

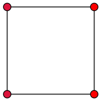

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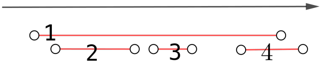

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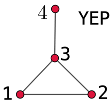

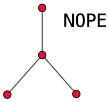

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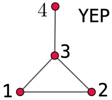

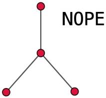

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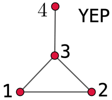

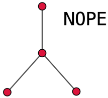

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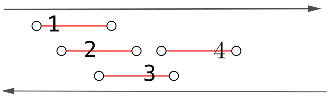

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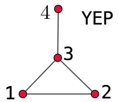

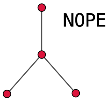

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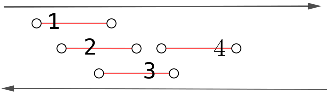

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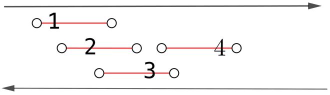

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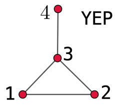

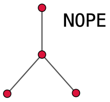

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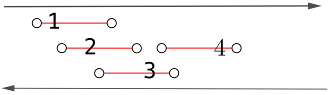

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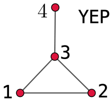

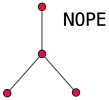

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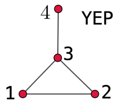

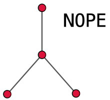

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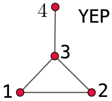

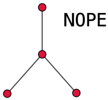

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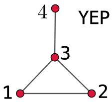

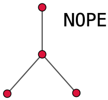

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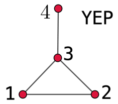

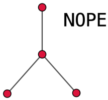

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Cluster graphs are exactly the "P_3-free graphs", i.e. the graphs without any induced three-vertex path.

Long paths aren't cluster, though they are unit-interval.

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For LP lovers: interval matrices $A \in \{0,1\}^{m \times n}$ are totally unimodular (by induction on no. of rows). So the polytope

$$\{\mathbf{x} \in \mathbb{R}^n \text{ such that } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$$

has all vertices with all cooordinates in $\mathbb{N}$, for any $\mathbf{b}$ in $\mathbb{Z}^m$.

A new algebraic perspective:

Herzog–Hibi–Hreinsdottir–Rauh–Kahle (2009) introduced the following correspondence:
Graph G with e edges, n vertices $\rightsquigarrow$ **binomial edge ideal**

$$(x_i y_j - x_j y_i) \ : \ ij \text{ is an edge in } G.$$

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1. For **any** graph, this ideal is radical.
2. A graph is unit-interval $\iff$ the generators of its BEI form a (squarefree) Gröbner basis.

And several exciting developments, e.g. Matsuda (2017) showed that if a graph is weakly-closed, then the quotient by its BEI is F-pure in characteristic p; Seccia in her thesis (2022) proved that a graph is weakly-closed if and only if its BEI is a Knutson ideal.

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II. Simplicial complexes.

Conventions

We write d-faces by listing vertices in increasing order, i.e. if we write $F = [a_0, a_1, \cdots, a_d]$, we mean $a_0 < a_1 < \ldots < a_d$. So $\min F = a_0$ and $\max F = a_d$.

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Σ_n^d is the d-skeleton on the $(n-1)$-dimensional simplex with vertex set $\{1, \ldots, n\}$.

Emtander's 2010 definition:

$\exists$ labeling such that for each d-face $F = [a_0, a_1, \cdots, a_d] \in \Delta$...

for any facet G of Δ with $\max F = \max G$, the complex Δ contains the full d-skeleton of the simplex on the vertex set $F \cup G$.

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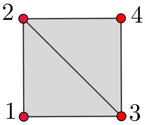

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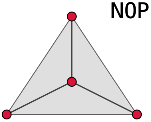

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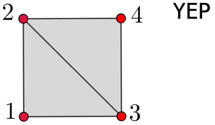

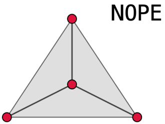

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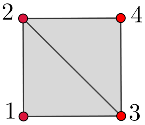

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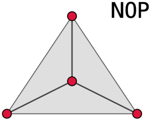

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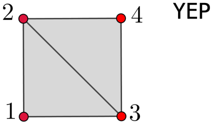

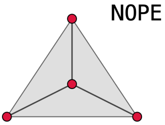

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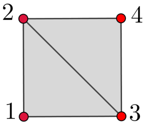

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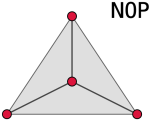

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Deleting even-labeled vertices $\rightsquigarrow$ a length-t (induced) cycle.

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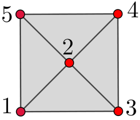

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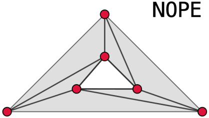

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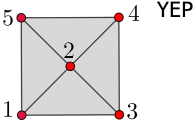

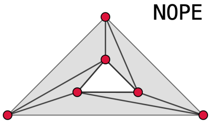

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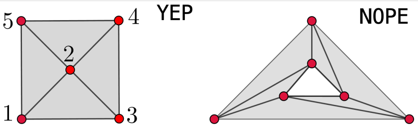

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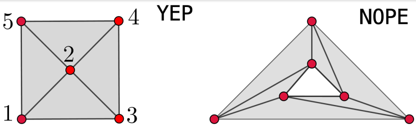

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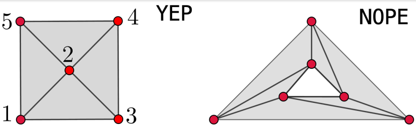

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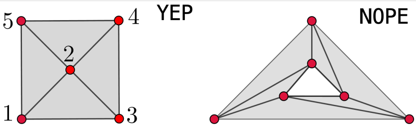

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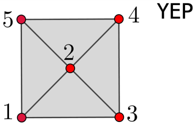

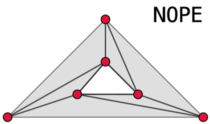

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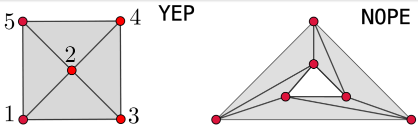

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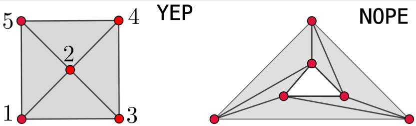

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1. (i) either Δ contains all d-faces $G \leq F$ with $\min G = \min F$,
2. (ii) or Δ contains all d-faces $H \geq F$ with $\max H = \max F$.

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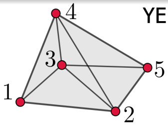

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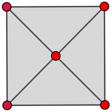

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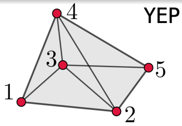

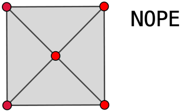

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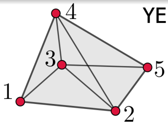

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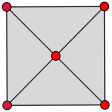

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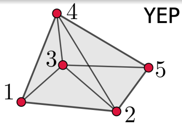

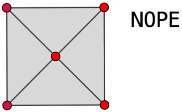

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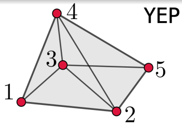

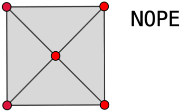

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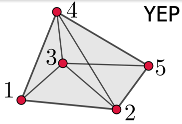

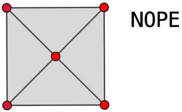

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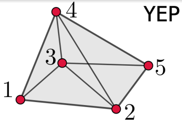

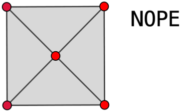

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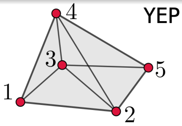

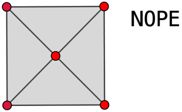

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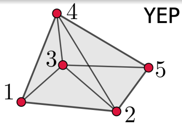

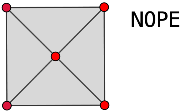

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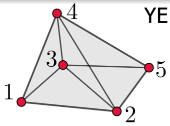

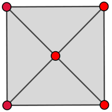

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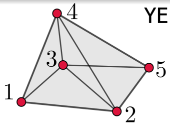

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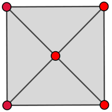

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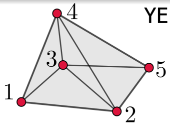

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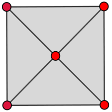

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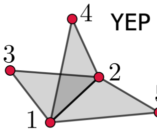

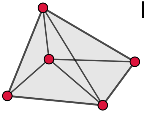

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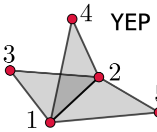

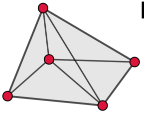

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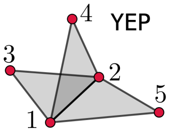

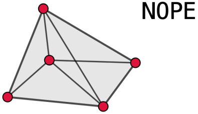

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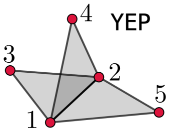

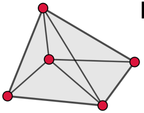

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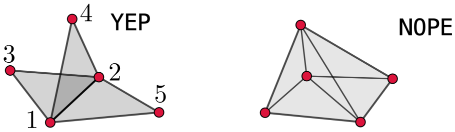

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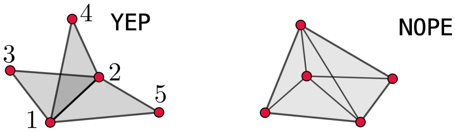

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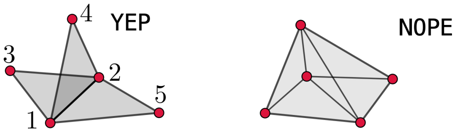

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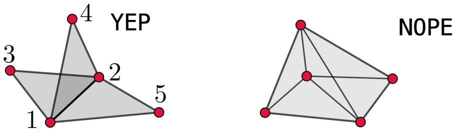

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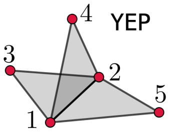

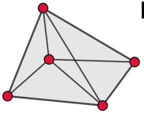

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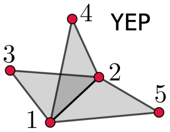

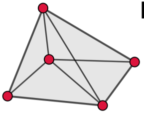

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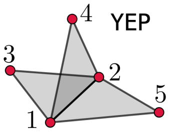

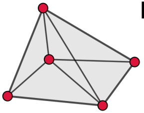

Generalizes interval graphs, passes to the 1-skeleton.

Theorem (BB-Seccia–Varbaro 21+)

For $d \geq 1$, $S^d \overset{\text{def}}{=} \text{susp}(\Sigma_d^{d-1})$, aka the boundary of the $(d+1)$-complex of facets H_1, H_2, is semi-closed, but not interval.

(The given labeling assigns labels 1 and $d+3$ to the apices; it's semi-closed, not interval. If either 1 or $d+3$ is not assigned to an apex, some d-face H contains both 1 and $d+3$; were the labeling interval, the 3 facets $[1, \ldots, d] * i$, $i = d+1, d+2, d+3$, would be in S^d, a contradiction with S^d manifold.)

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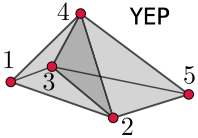

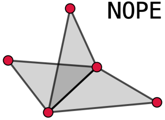

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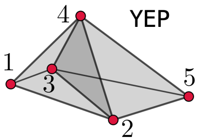

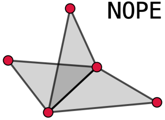

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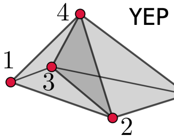

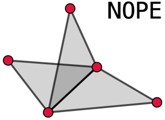

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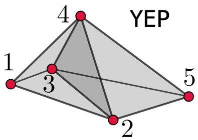

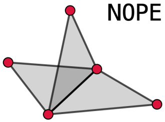

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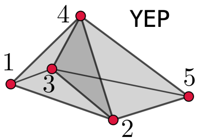

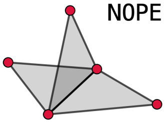

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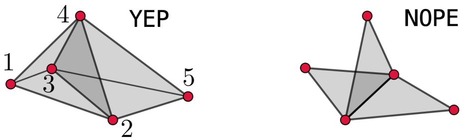

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which suggests how to relabel the vertices.

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Ideal generated by f polynomials, each sum of $(d+1)!$ squarefree monomials of degree $d+1$, in a ring with $(d+1)n$ variables.

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Let Δ be a strongly-connected d-complex.
Δ is unit-interval $\iff$ Δ is traceable and with respect to the same labeling, the generators of its DFI form a Gröbner basis wrt LEX.

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