Complexes from point clouds

Geometry, topology, algebra, and combinatorics

Ulrich Bauer

TUM

Mar 22, 2022

Combinatorial Coworkspace 2022

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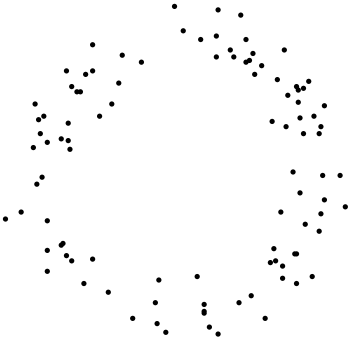
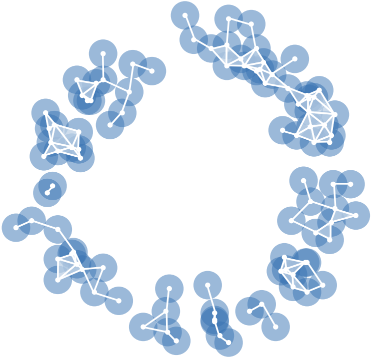
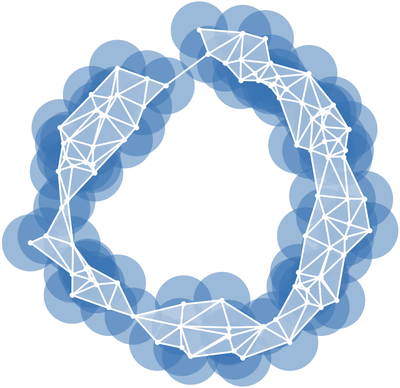
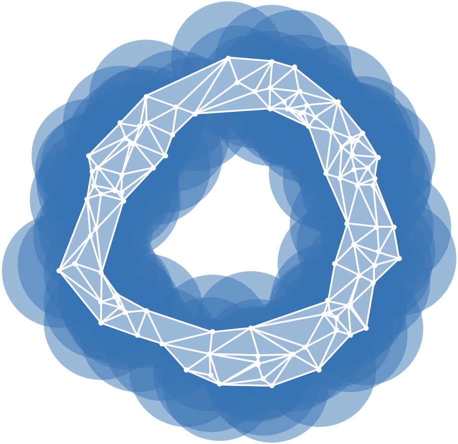
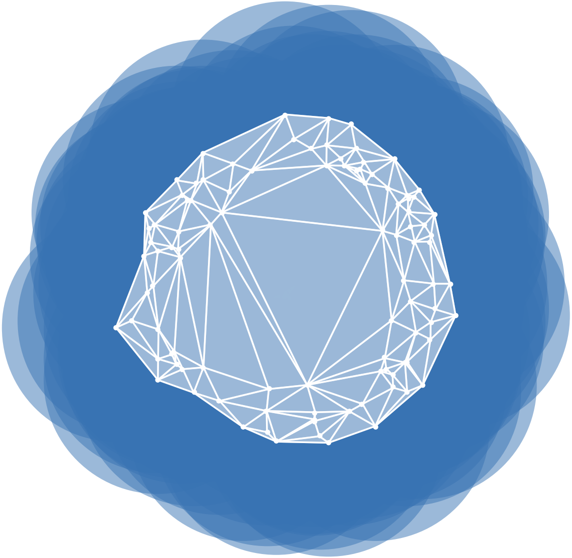


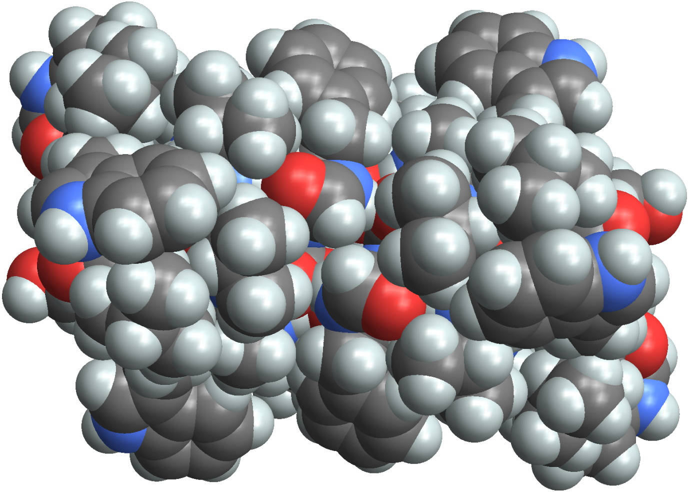








Geometry and topology of biomolecules



Gramicidin (an antibiotic functioning as an ion channel)

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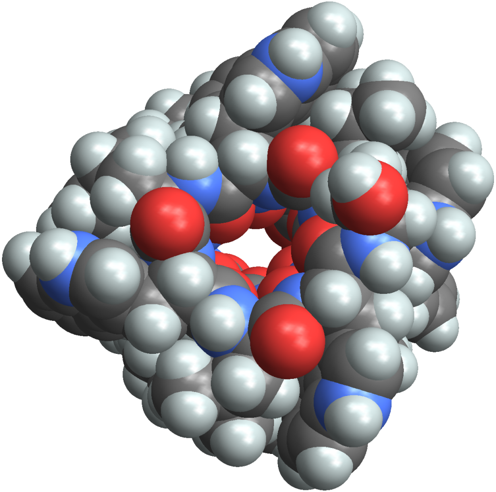

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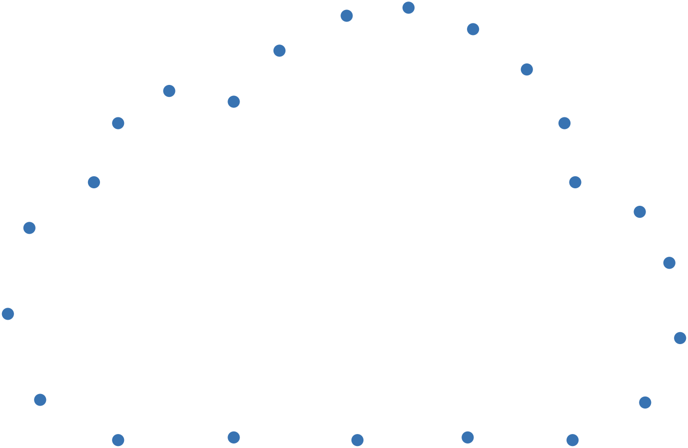
Čech complexes



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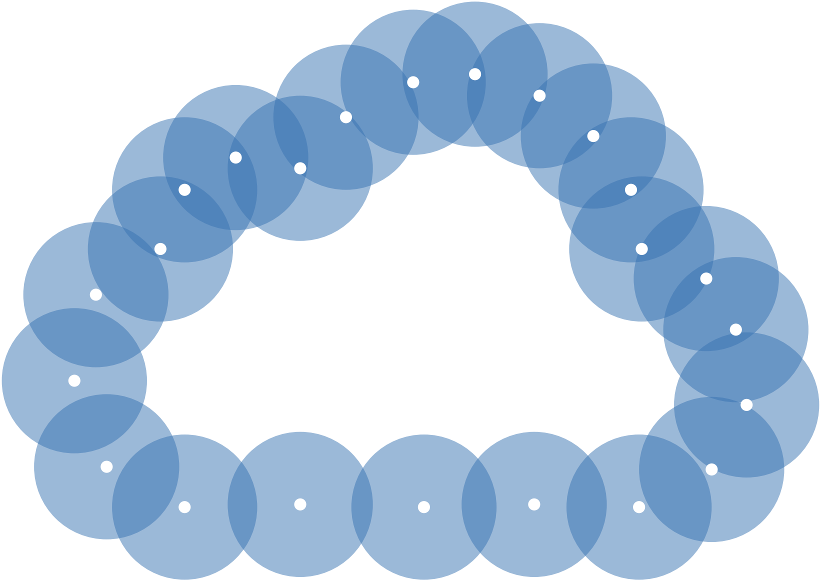

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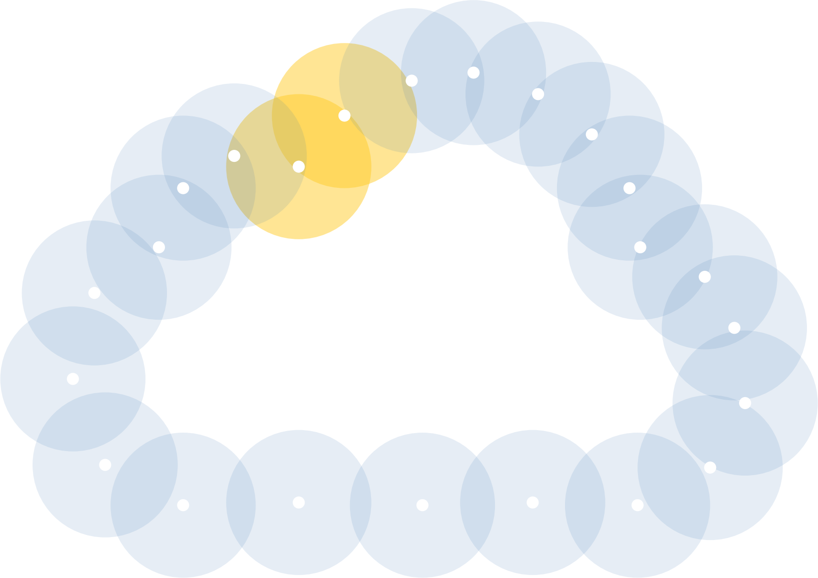

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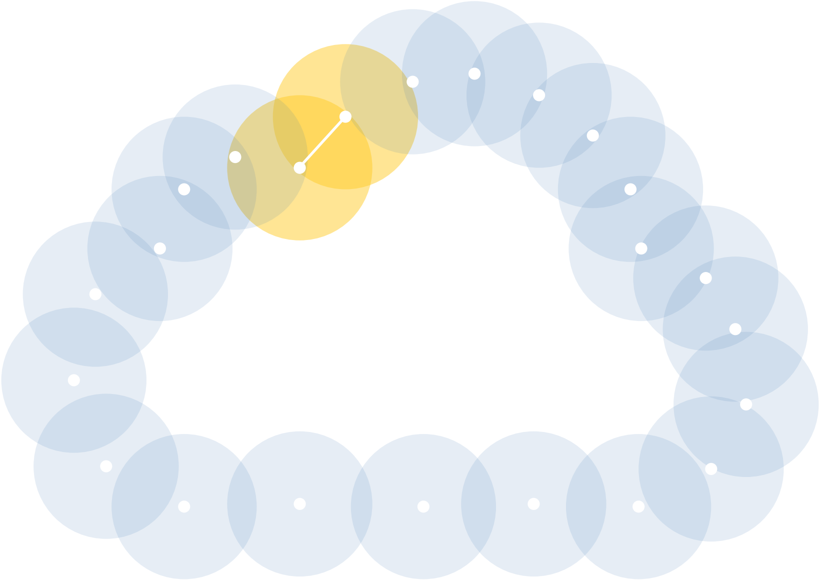

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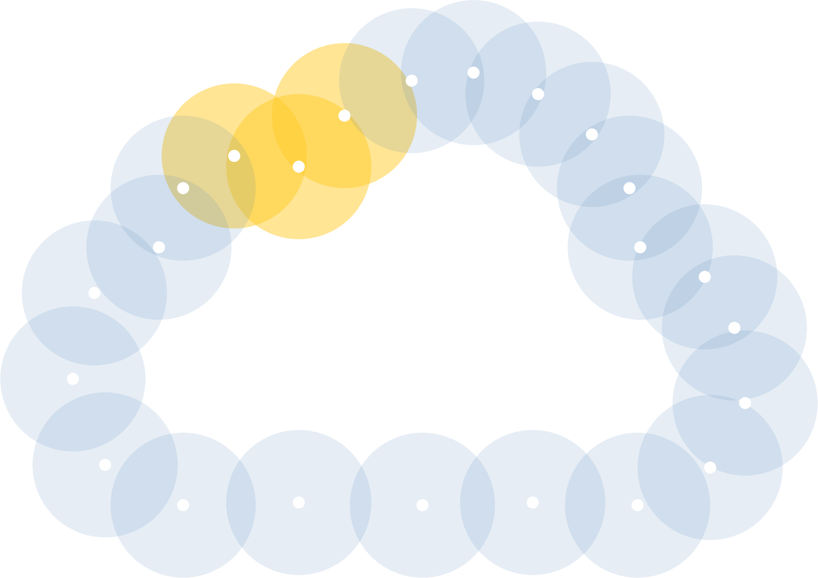

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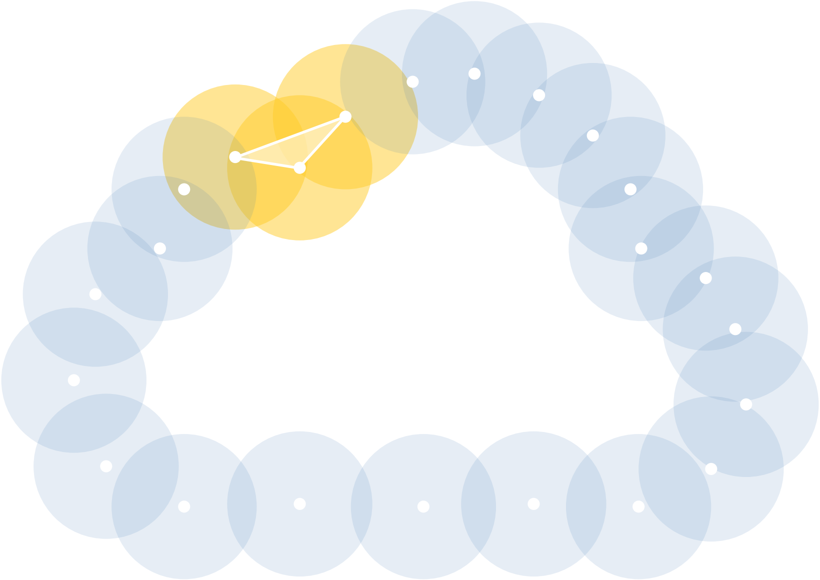

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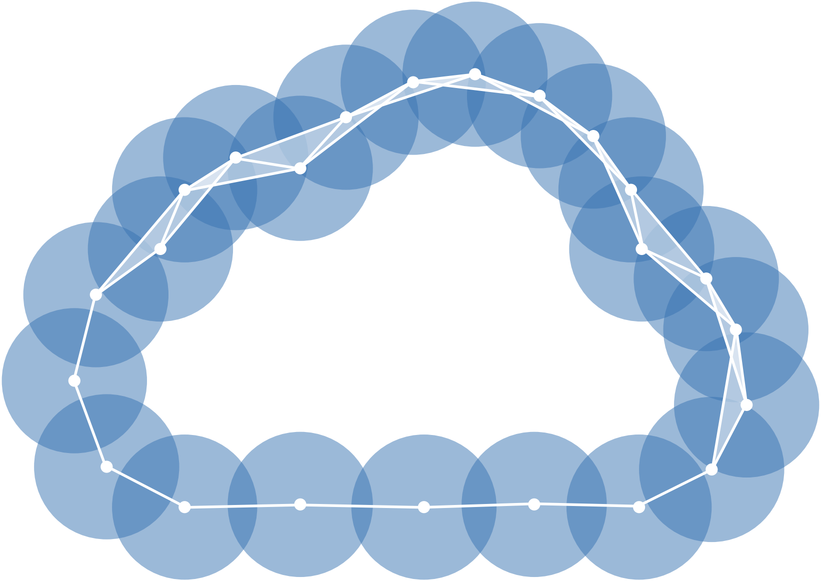

Definition

Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \left\{ J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset \right\} .$$

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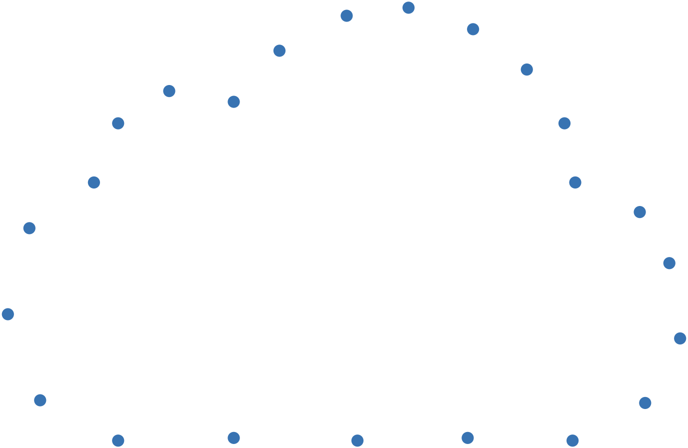
A Unified View on the Functorial Nerve Theorem and its Variations
Preprint, arXiv:2203.03571, 2022

Delaunay complexes



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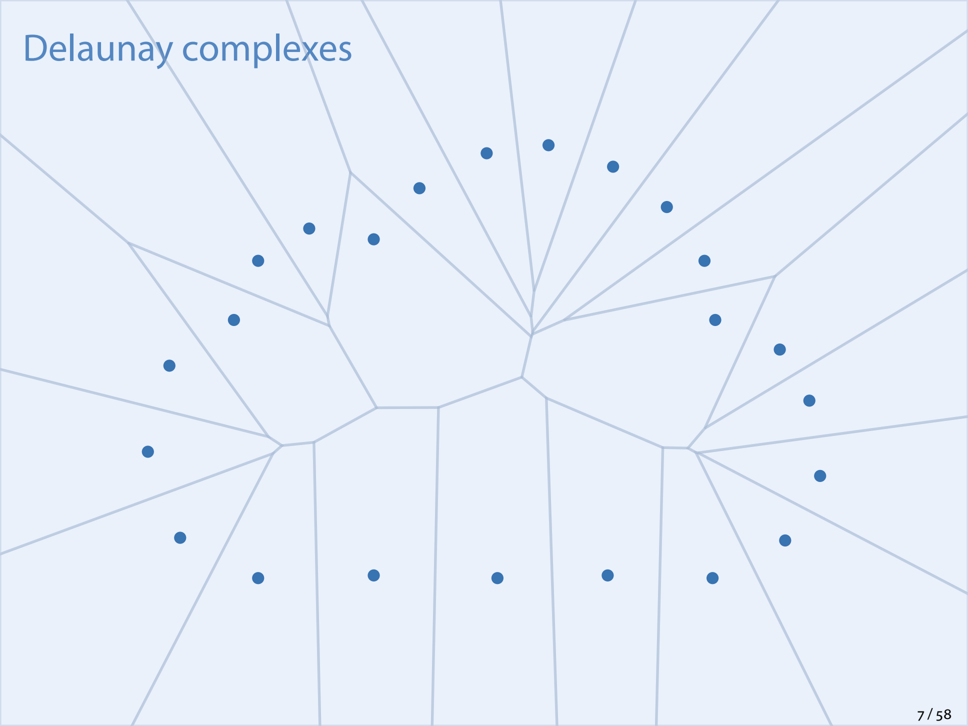

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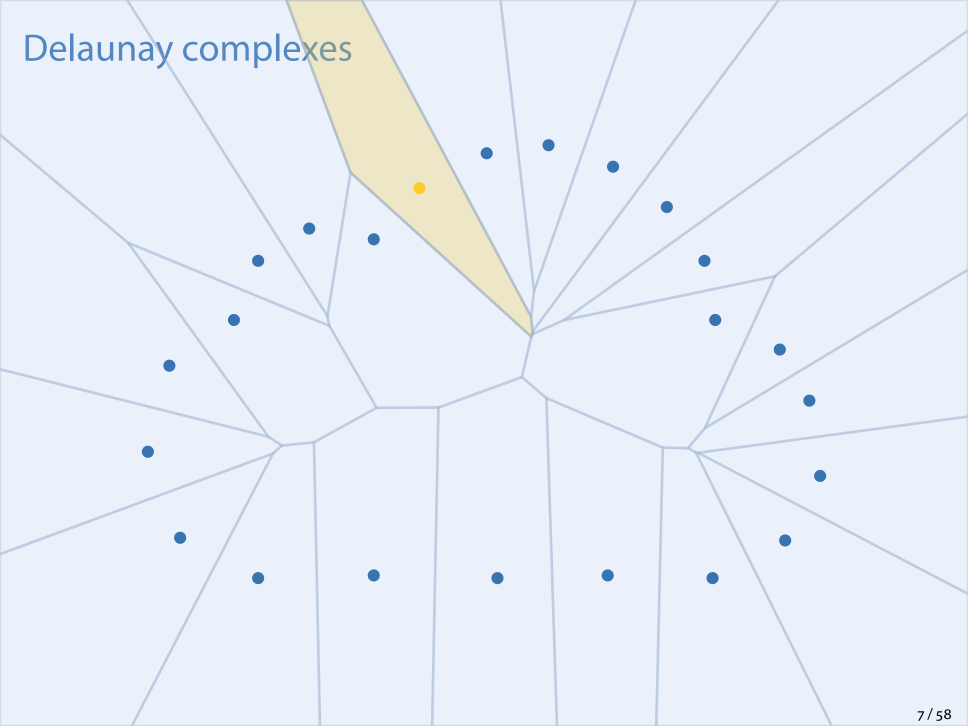

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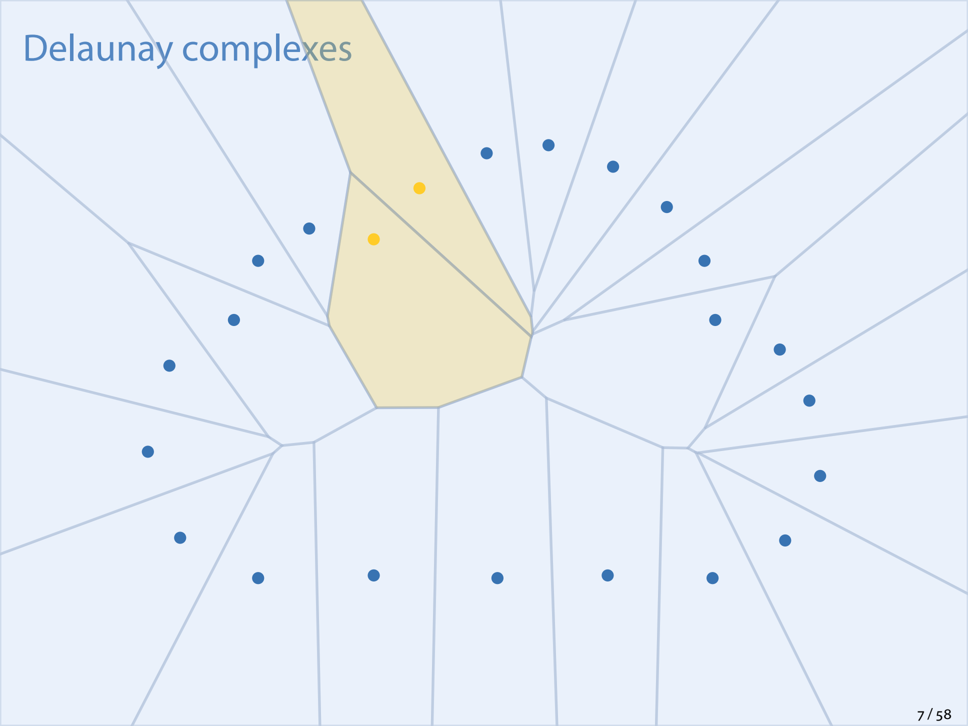

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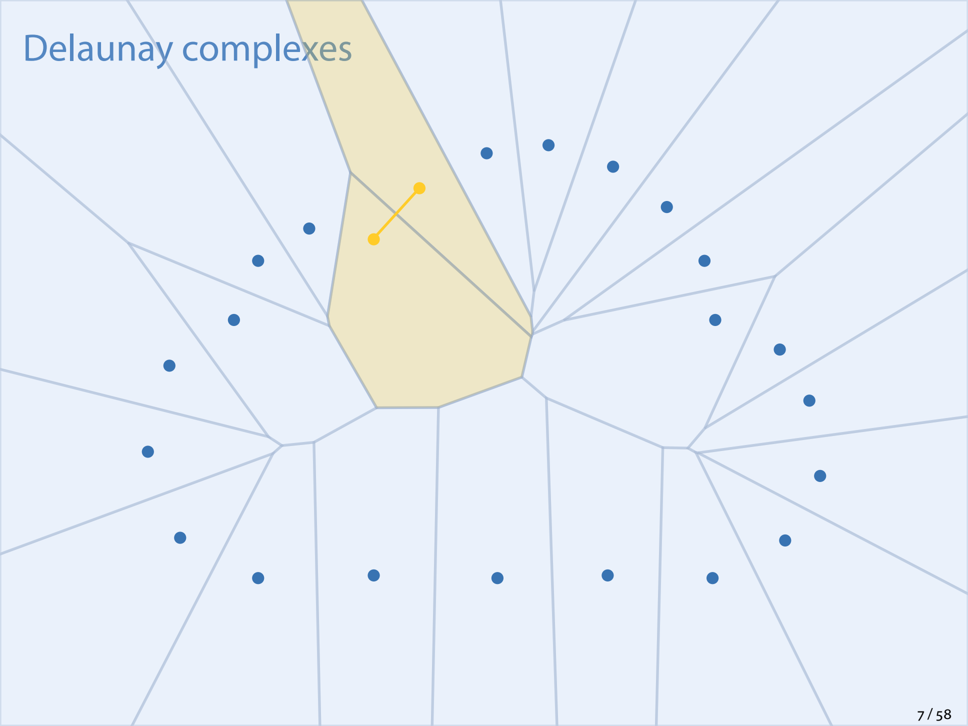

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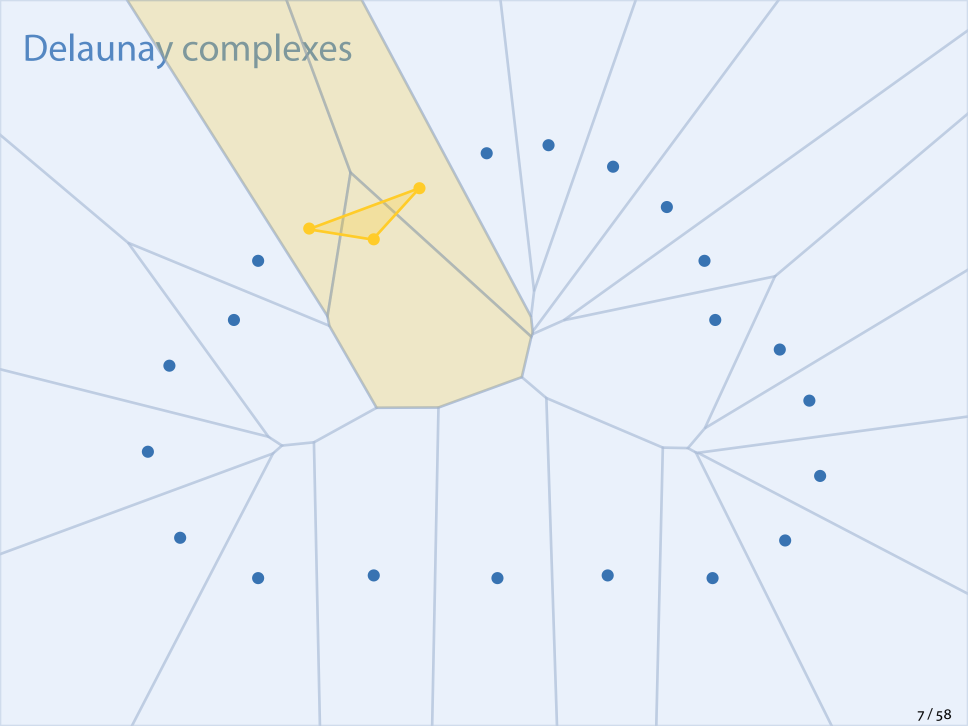

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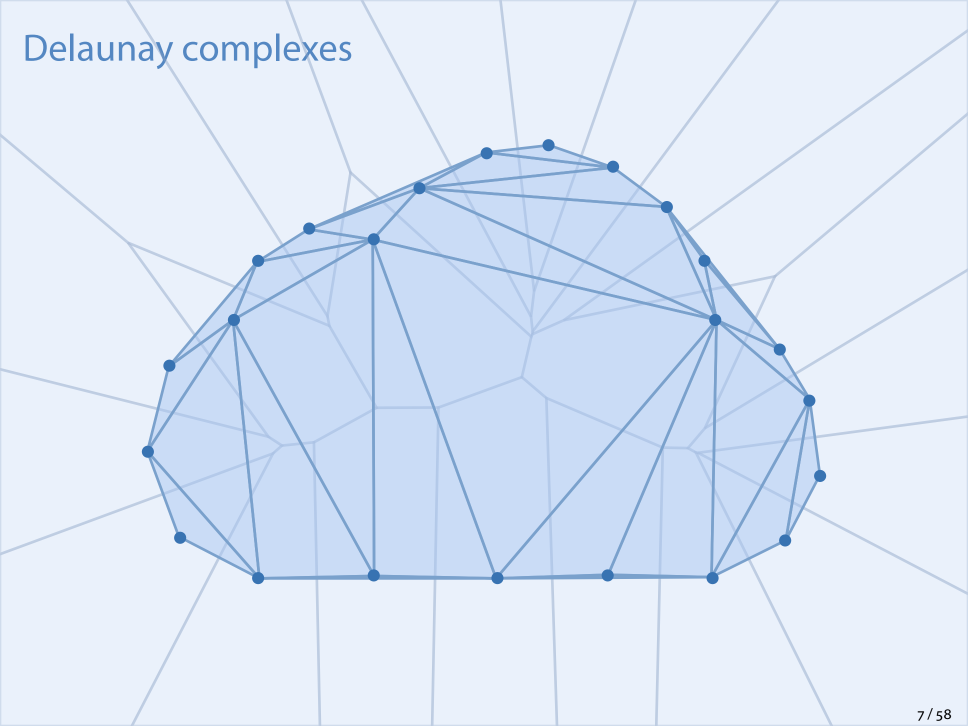

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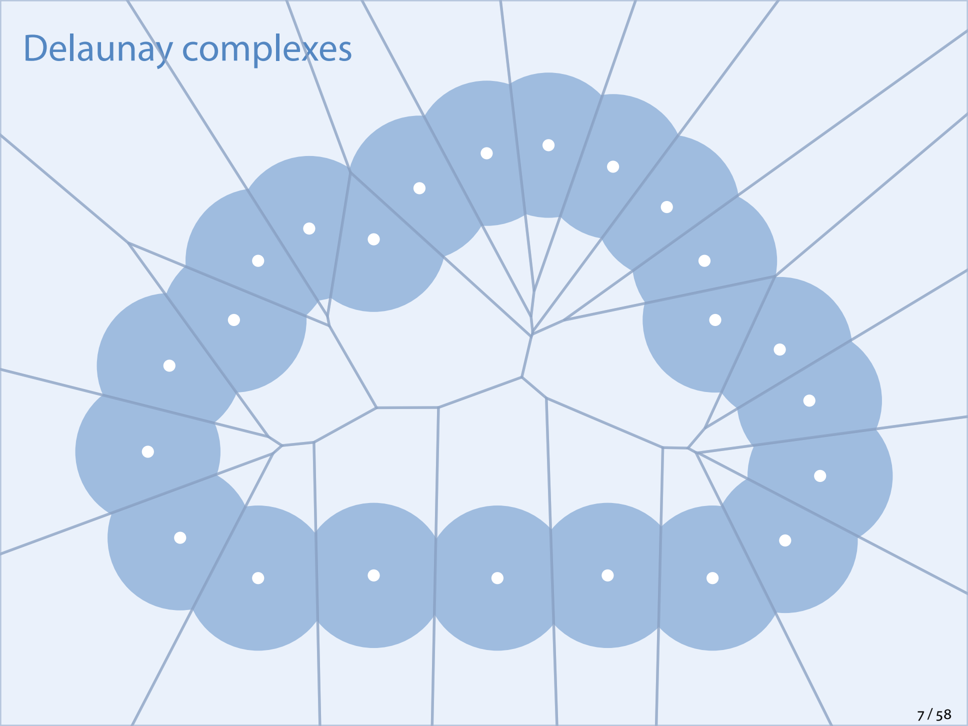

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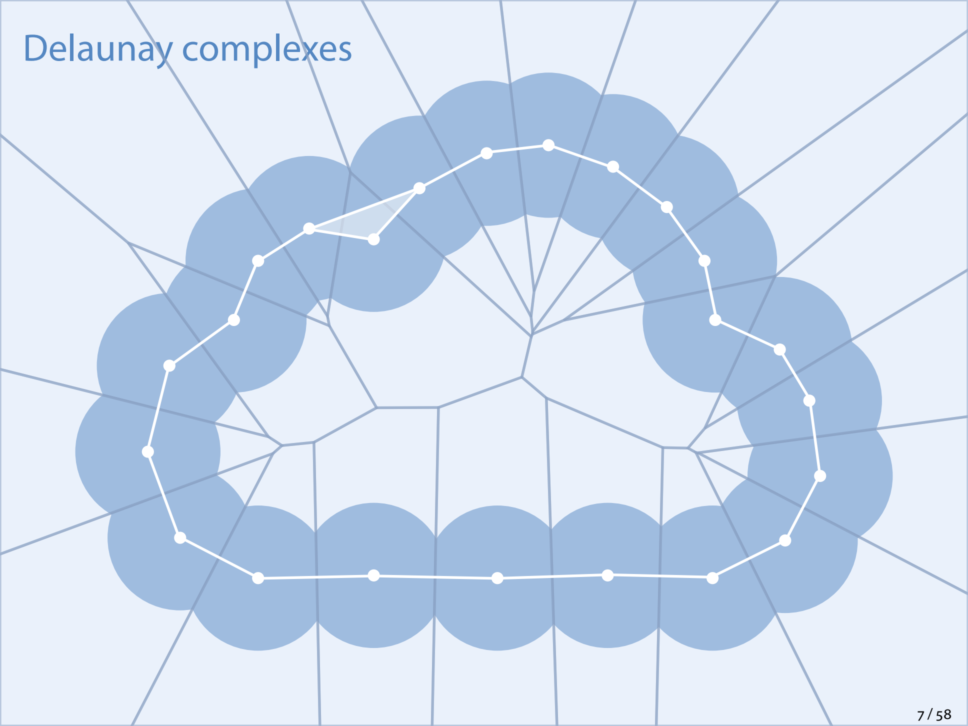

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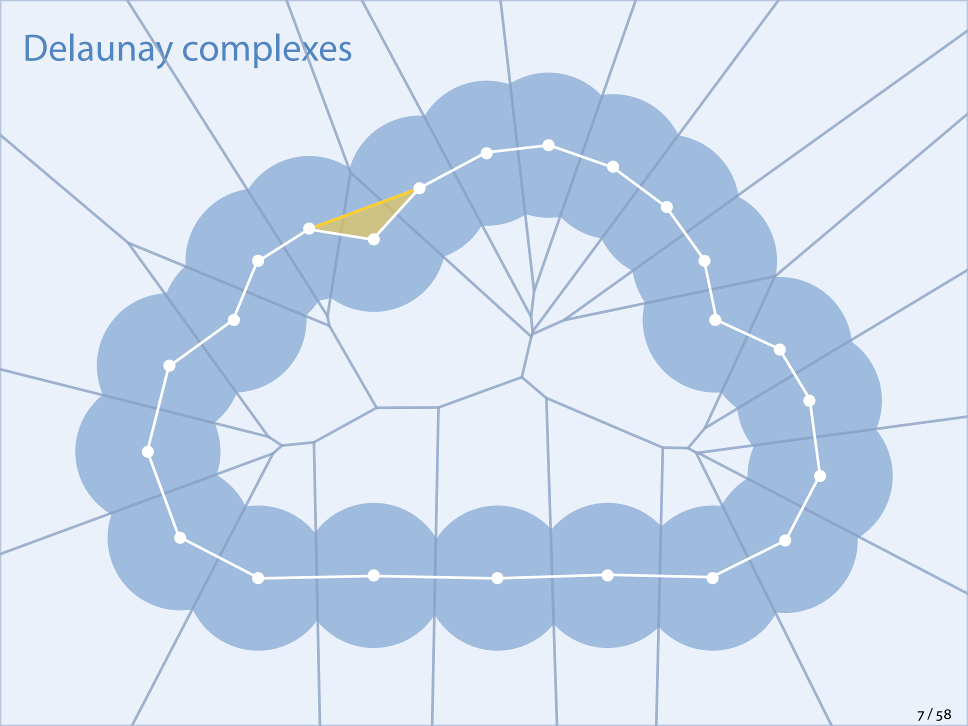

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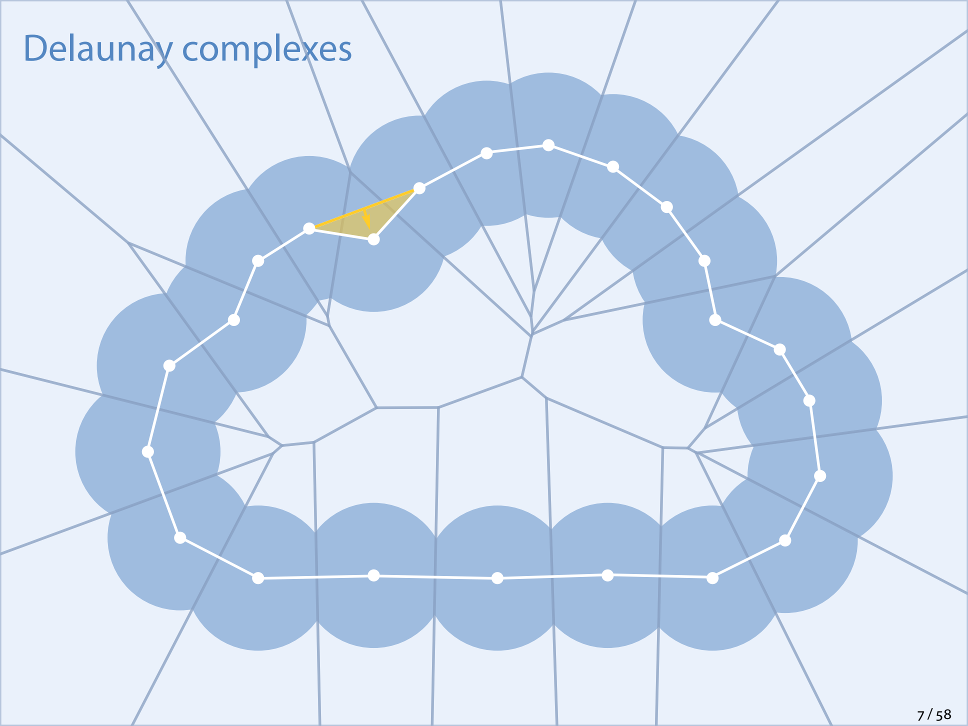

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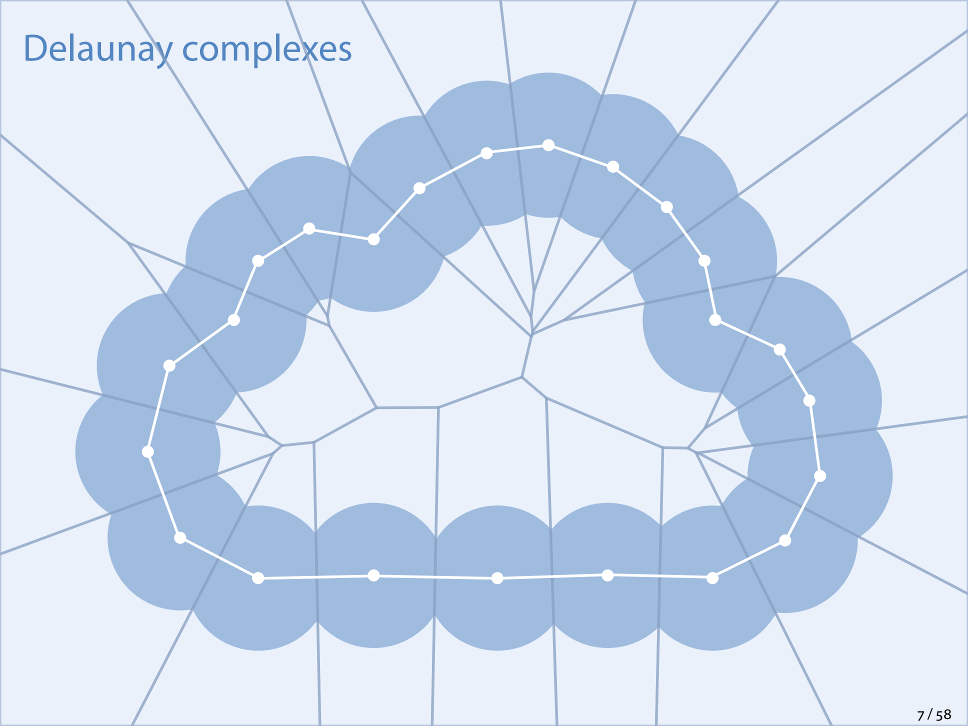
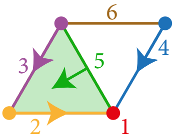


Discrete Morse theory



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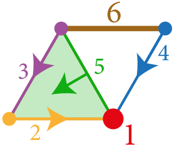

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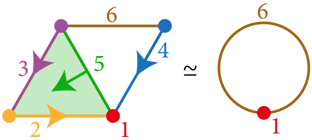
Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a CW complex build from the critical simplices of f.



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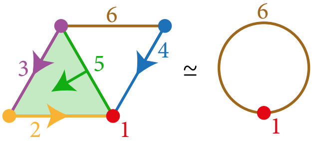
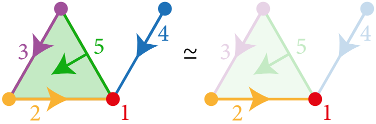


Discrete Morse functions – and their gradients – encode collapses of sublevel sets:



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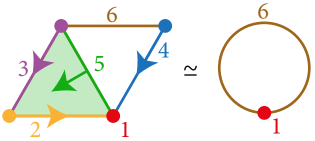

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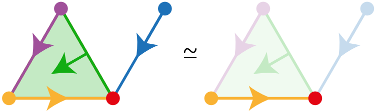
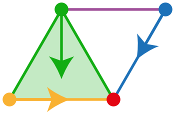


Generalized discrete Morse theory

Definition (Forman 1996, Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into *intervals* of the face poset:

$$[L, U] = \{Q \mid L \subseteq Q \subseteq U\}$$



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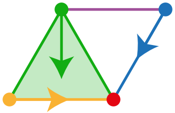

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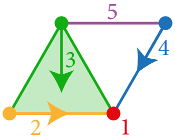
A *generalized discrete Morse function* $f : K \to \mathbb{R}$ satisfies:

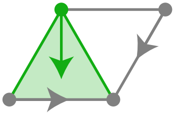


- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes (for all $t \in \mathbb{R}$)
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)

Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical face interval $[L, U] \in V$:



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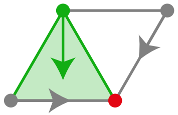

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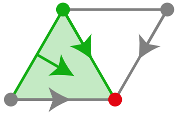
- partition $[L, U]$ into pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

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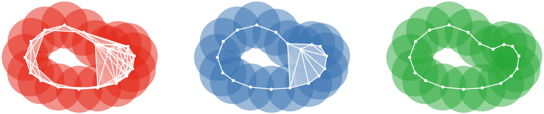
Theorem (B., Edelsbrunner 2017)

Čech, Delaunay, and Wrap complexes are related by collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X,$$

encoded by a single discrete gradient field.



Delaunay and Wrap complexes

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Homology inference

Inferring homology from samples

Given: finite sample $P \subset X$ of unknown shape $X \subset \mathbb{R}^d$

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This can work, but requires strong assumptions:

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let X be a submanifold of $\mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

- P_δ *covers* X*, and*
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Then $H_*(X) \cong H_*(P_{2\delta})$.

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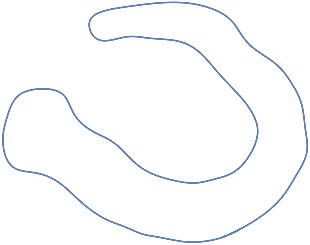

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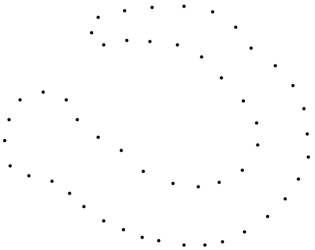

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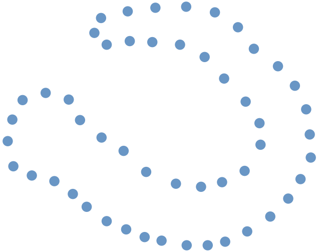

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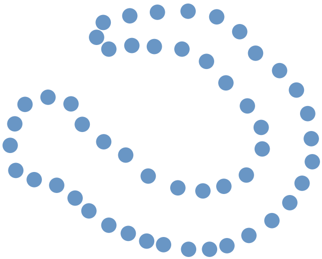

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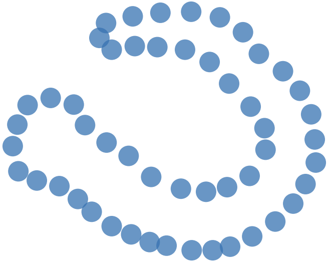

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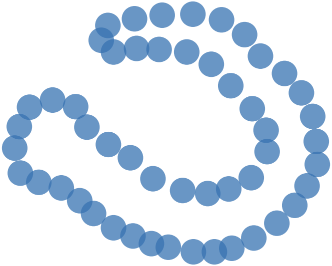

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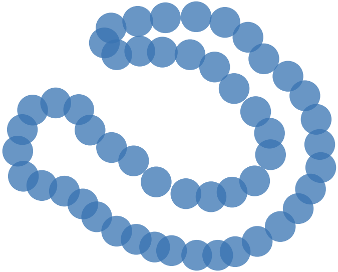

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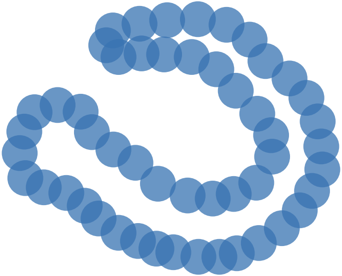

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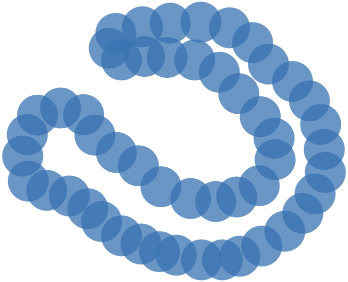

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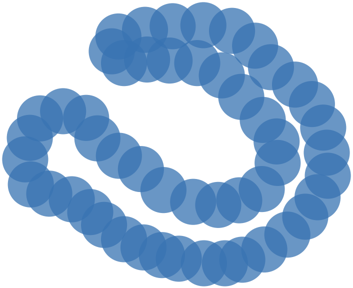

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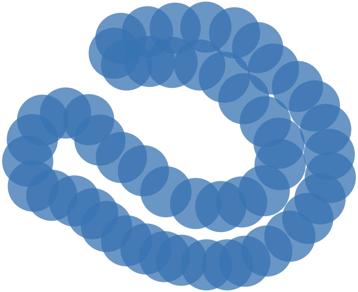

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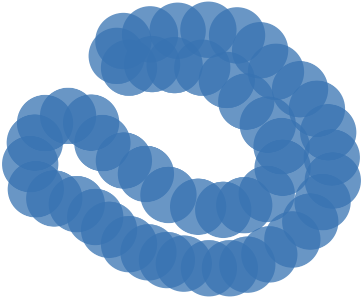

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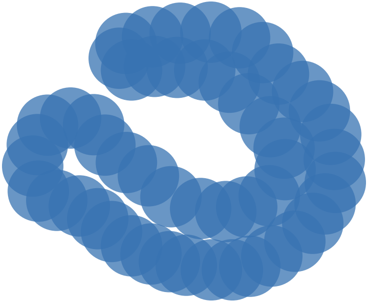

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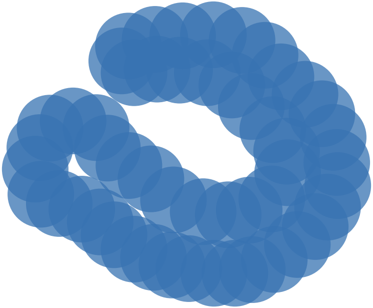

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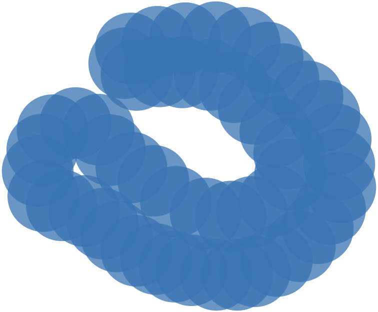

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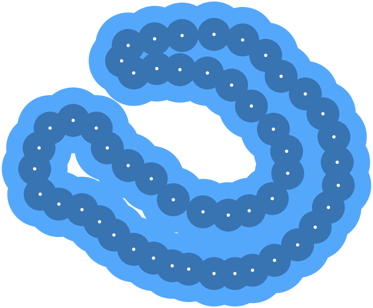

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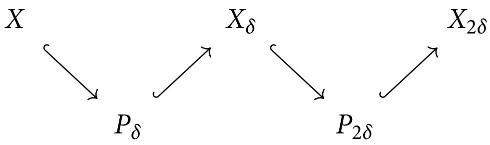

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& \searrow & & \nearrow & & \searrow & & \nearrow & \\
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\end{array}
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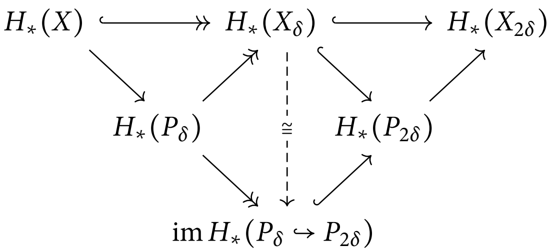

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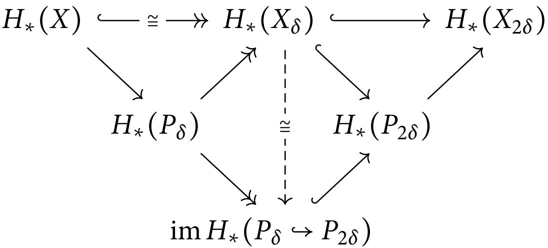
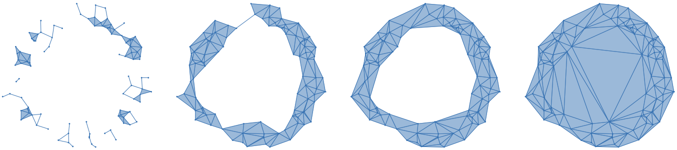


□

What is persistent homology?



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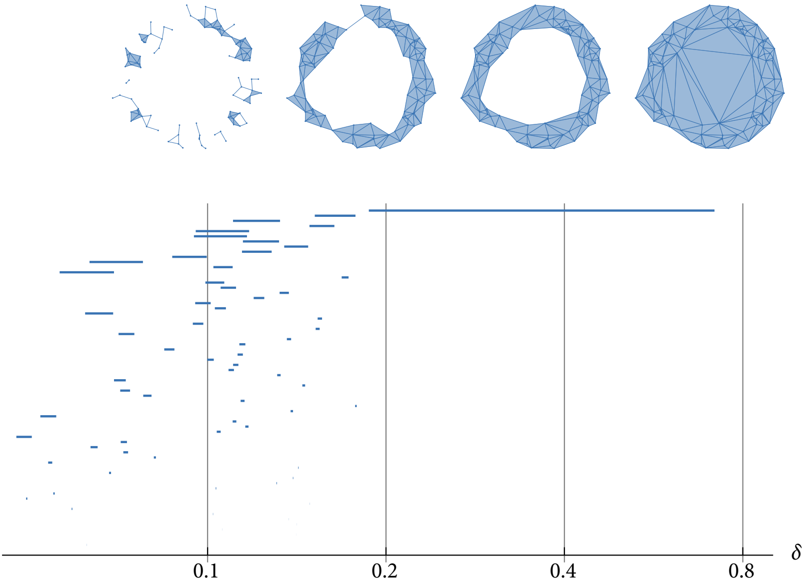

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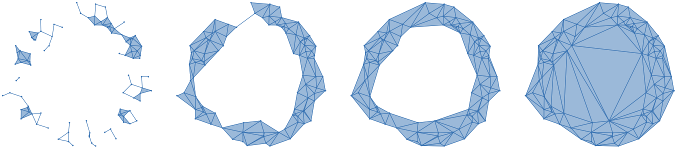

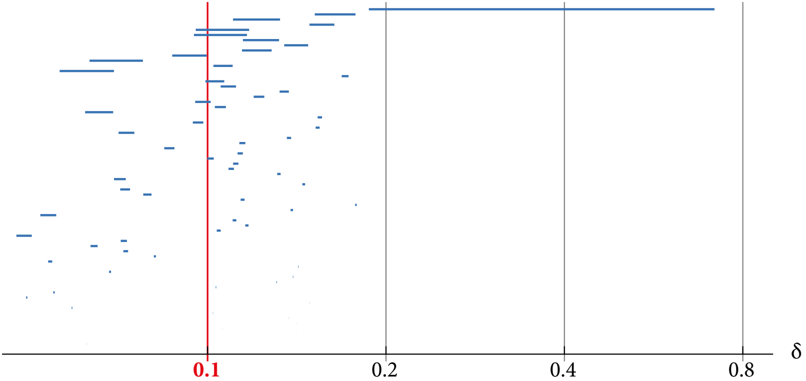

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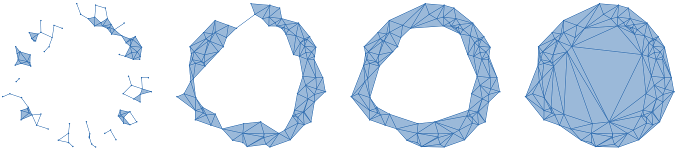

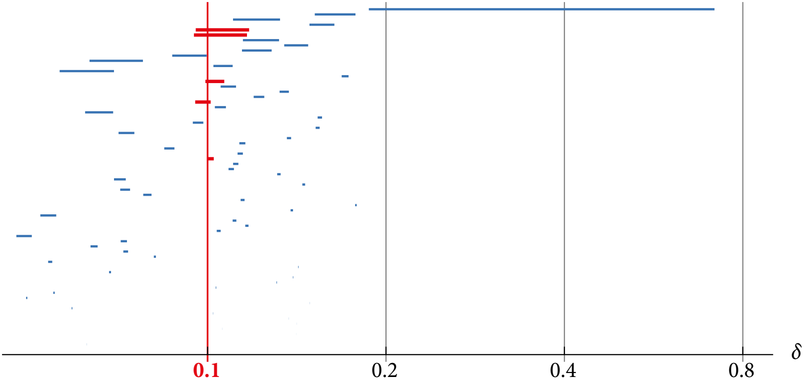

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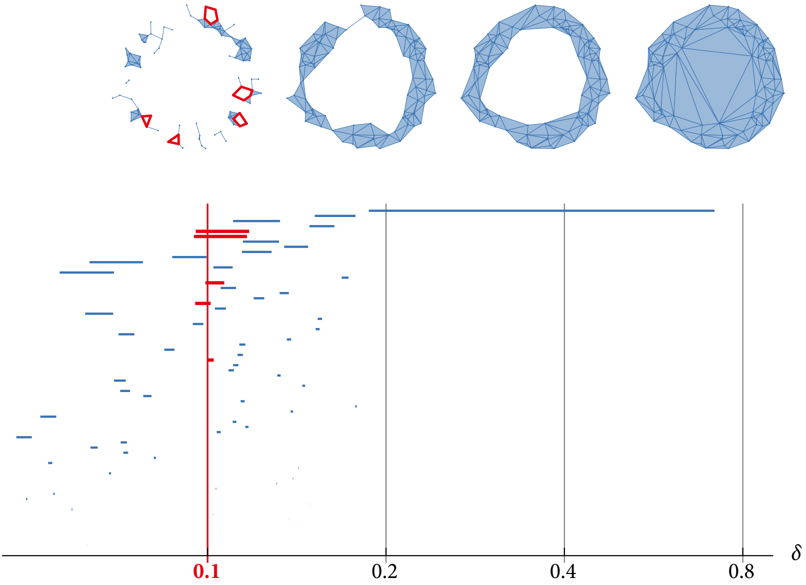

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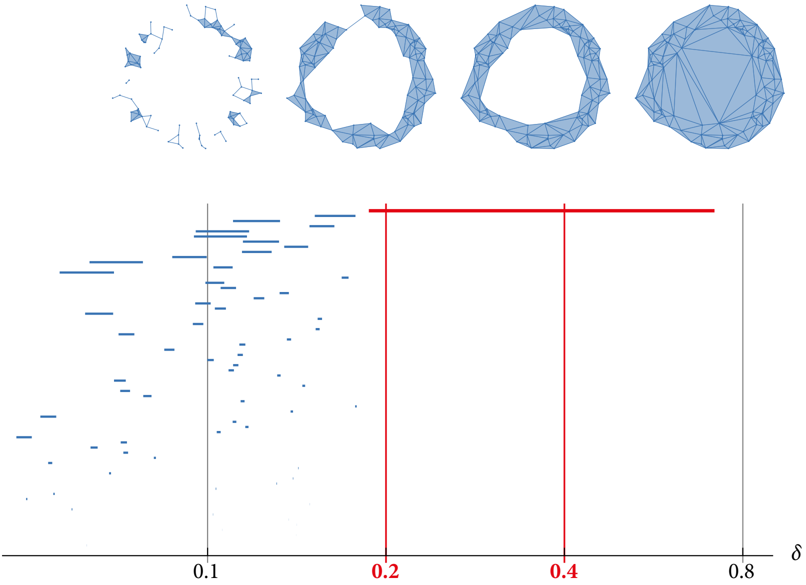

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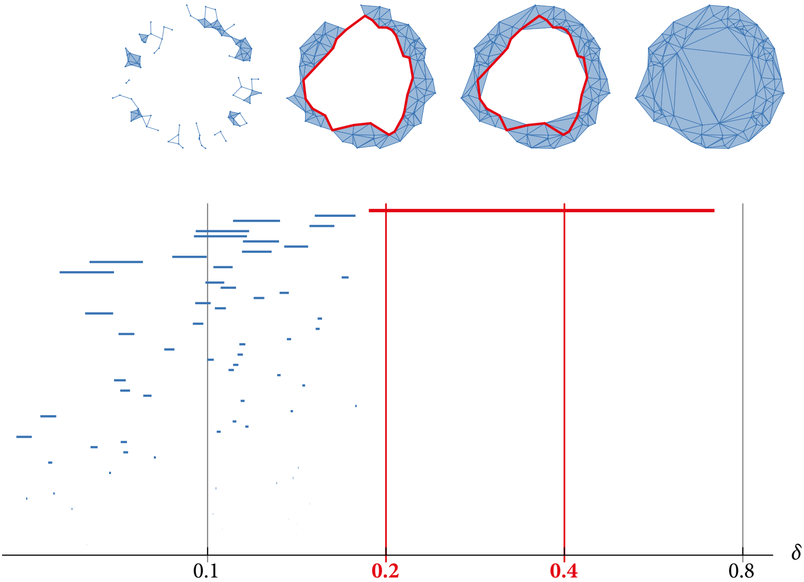

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Persistent homology is the homology of a filtration.

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$ of topological spaces, indexed over the poset of real numbers $\mathbf{R} := (\mathbb{R}, \leq)$

$$\cdots\cdots\to K_s \hookrightarrow K_t \cdots\cdots\to$$

  - a topological space K_t for each $t \in \mathbb{R}$
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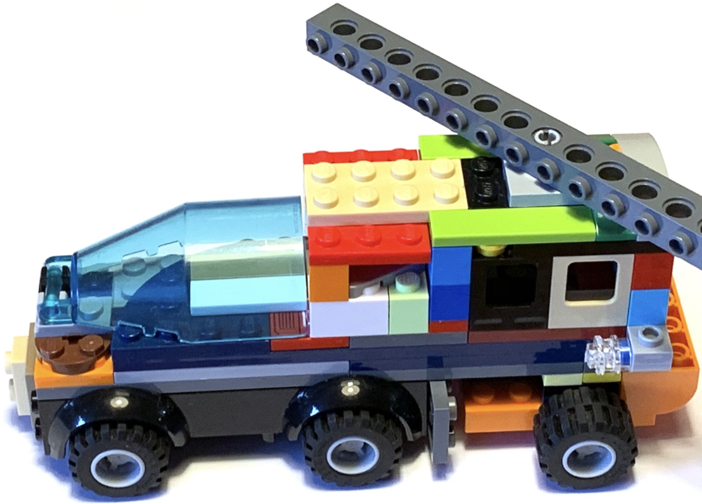
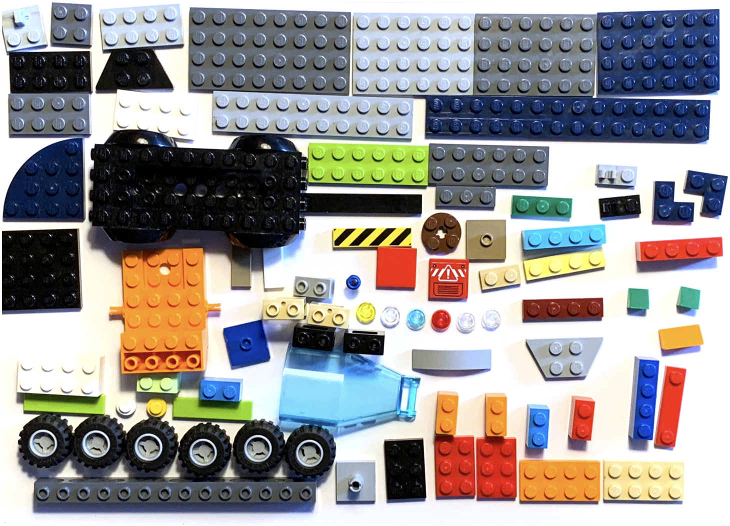
- Apply homology $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M = H_* \circ K : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*:

$$\cdots\cdots\rightarrow M_s \longrightarrow M_t \cdots\cdots\rightarrow$$





Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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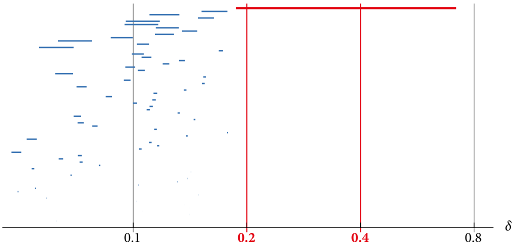
- The supporting intervals form the *persistence barcode*.



Computation

Homology by matrix reduction

Notation:

- D: boundary matrix (with $\mathbb{Z}_2$ coefficients)
- R_i: ith column of matrix R
- $\operatorname{pivot} R_i$: maximal row index with nonzero entry in column R_i

Matrix reduction algorithm (variant of Gaussian elimination):

- $R = D, V = I$
- while $\exists i < j$ with $\operatorname{pivot} R_i = \operatorname{pivot} R_j$
  - add R_i to R_j, add V_i to V_j

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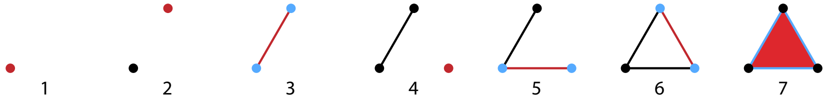
Result:

- $R = D \cdot V$ is reduced (each column has a unique pivot)
- V is full rank upper triangular

Persistent homology by matrix reduction



R:

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

$= D \cdot$

V:

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1		
6						1	
7							1

Algorithm:

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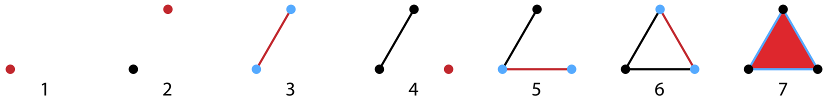

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1		
6						1	
7							1

V

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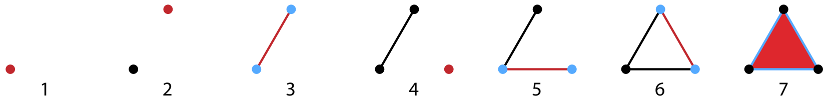

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1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

R

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	1	2	3	4	5	6	7
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7							1

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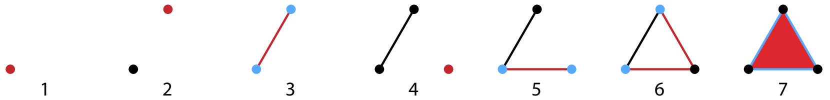

	1	2	3	4	5	6	7
1			1		1	1	
2			1			1	
3							1
4					1	0	
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
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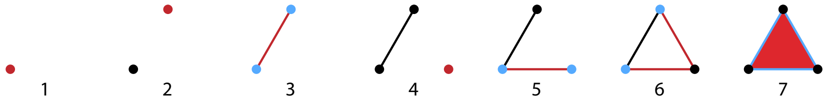

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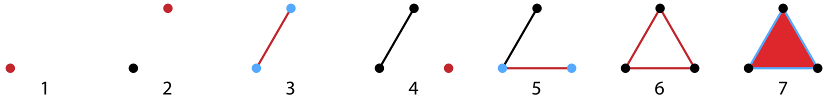

	1	2	3	4	5	6	7
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3							1
4					1		
5							1
6							1
7							

R

$= D \cdot$

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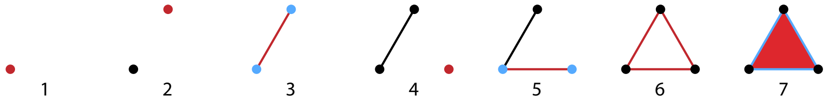

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$= D \cdot$

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2		1					
3			1			1	
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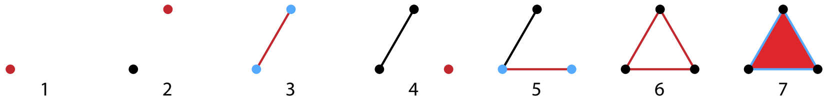
Persistent homology by matrix reduction



	1	2	3	4	5	6	7
1			1		1		
2			1				
3							1
4					1		
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
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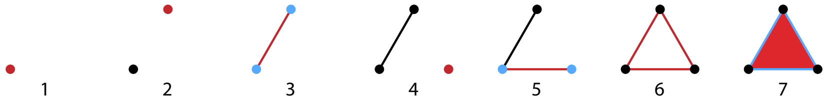

$R =$

	1	2	3	4	5	6	7
1			1		1		
2			1				
3							1
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6							1
7							

$= D \cdot V$, with $V =$

	1	2	3	4	5	6	7
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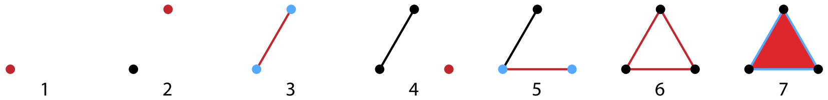

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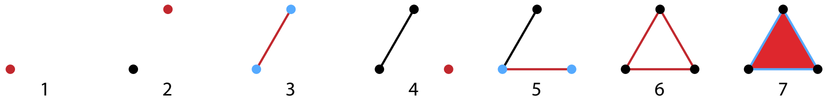

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Stability

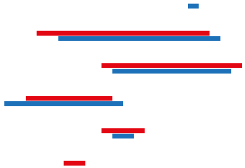
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $f, g : X \to \mathbb{R}$ *with* $\|f - g\|_\infty = \delta$ *(and some regularity assumptions).*

- *Consider the sublevel set filtrations* $f^{-1}(\infty, t]$ *and* $g^{-1}(\infty, t]$*, and*
- *take the resulting persistence barcodes.*

Then there exists a δ-matching *between the barcodes, meaning that:*



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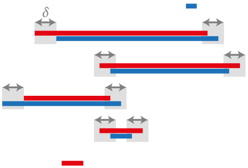

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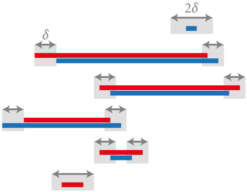

Persistence and stability: the big picture

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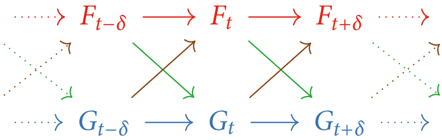

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$$\begin{array}{ccccccc} \cdots\to & F_{t-\delta} & \longrightarrow & F_t & \longrightarrow & F_{t+\delta} & \cdots\to \\ \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow \\ \cdots\to & G_{t-\delta} & \longrightarrow & G_t & \longrightarrow & G_{t+\delta} & \cdots\to \end{array} \qquad \forall t \in \mathbb{R}.$$

Applying homology, the persistence modules $H_*(F), H_*(G) : \mathbf{R} \to \mathbf{Vect}$ are δ-*interleaved:*

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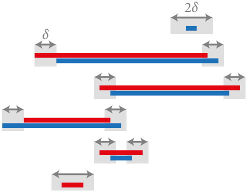
Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then there exists a δ-matching of their barcodes.



Structure of persistence sub-/quotient modules

Proposition (B, Lesnick 2015)

Let $M \twoheadrightarrow N$ be an epimorphism of persistence modules.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
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This construction is functorial.

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Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

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For $f : M \to N$ a general morphism of pfd persistence modules, the epi-mono factorization

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

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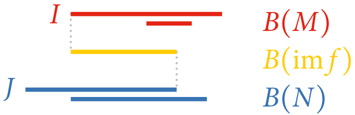

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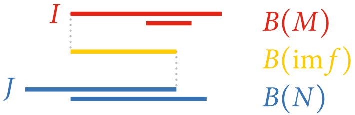


If f is a δ-interleaving morphism, then this is a δ-matching.

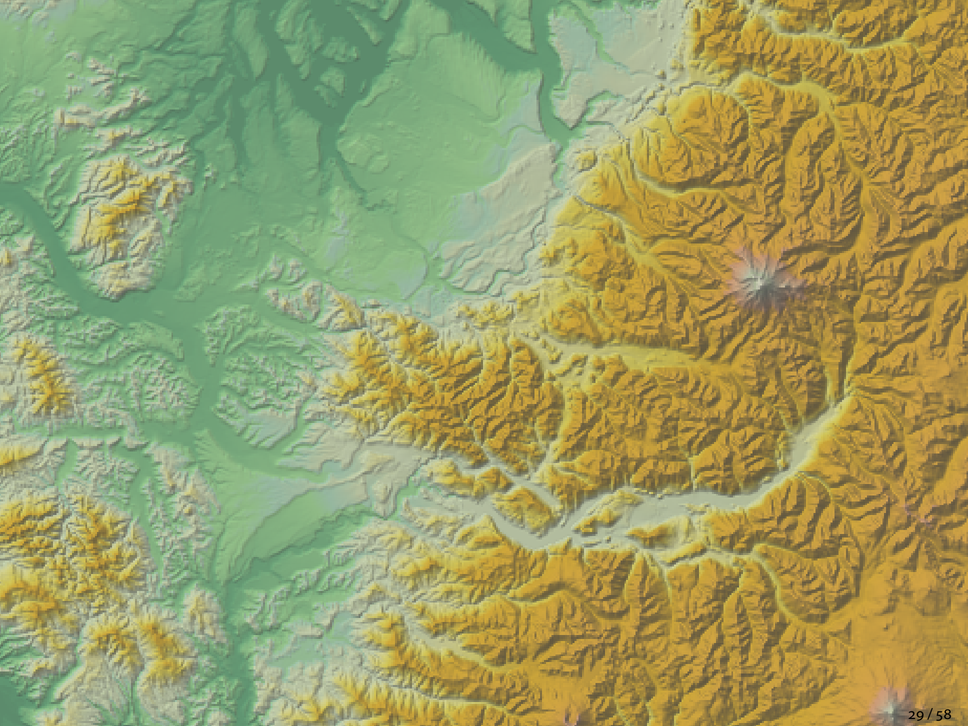
Simplification

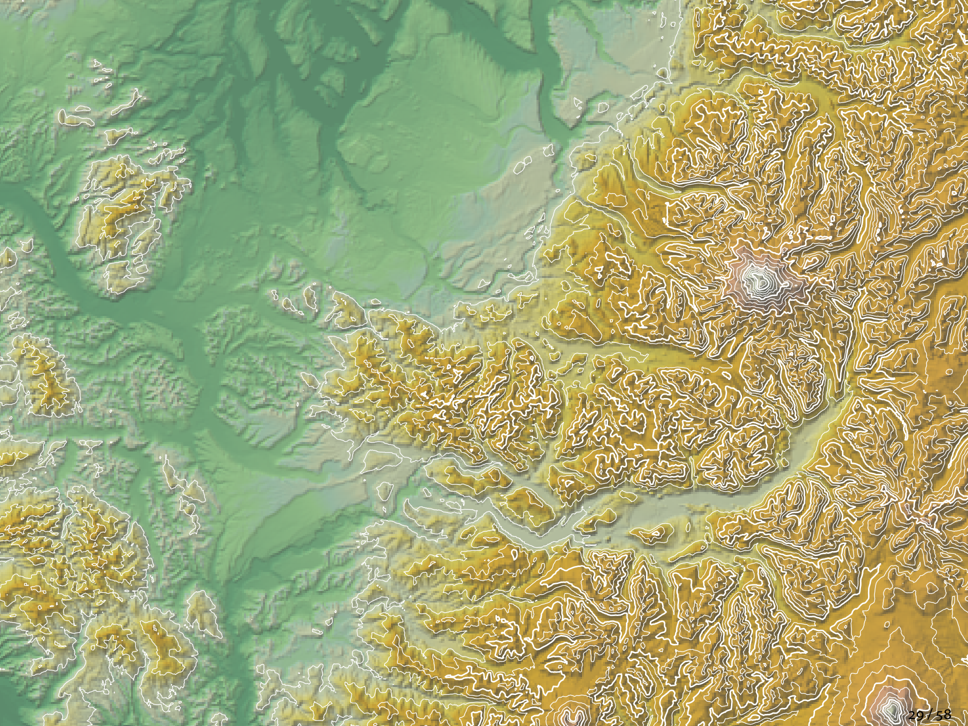
Topological simplification of functions

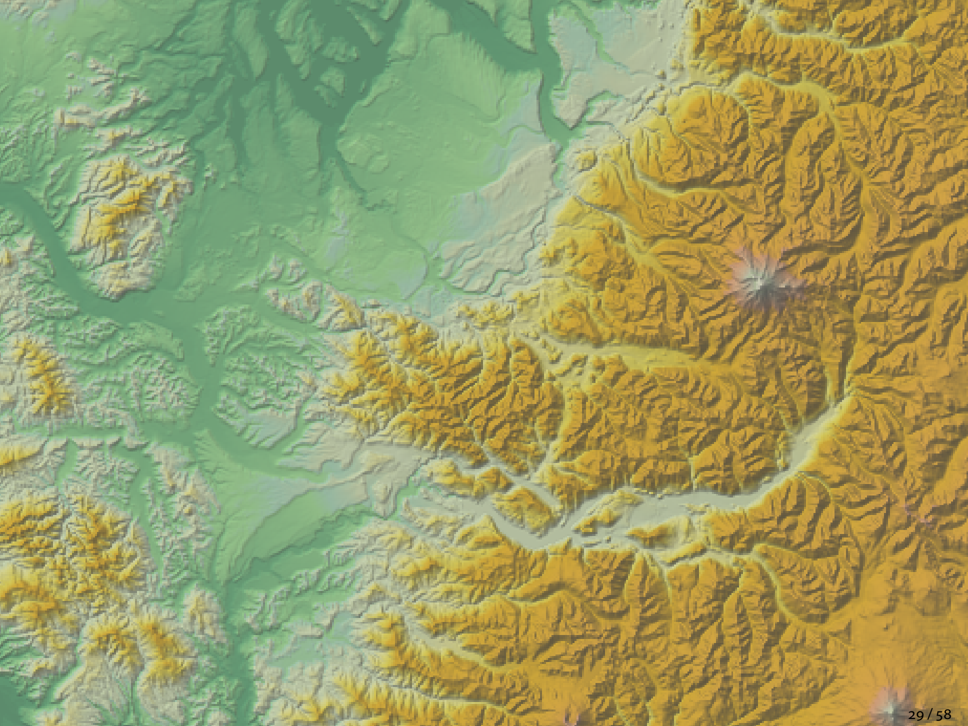
Consider the following problem:

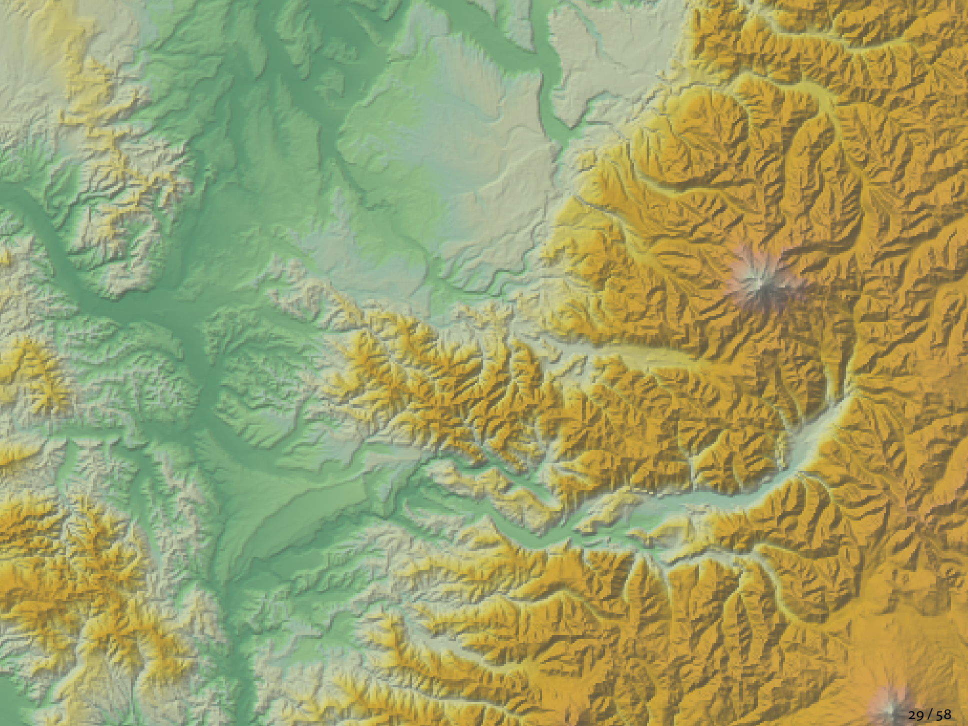
Problem (Topological simplification)

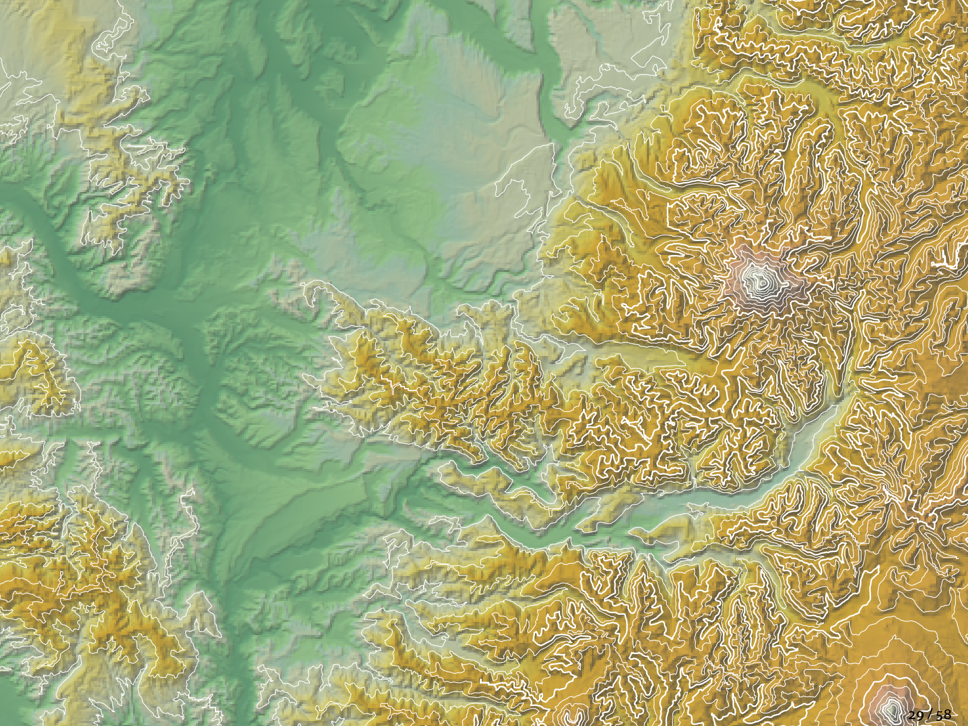
Given a function f and a real number $\delta \geq 0$, find a function f_δ with the minimal number of critical points subject to $\|f_\delta - f\|_\infty \leq \delta$.







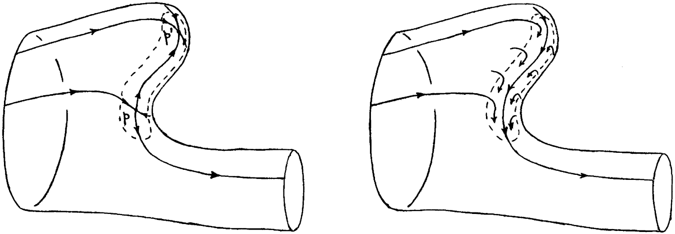




Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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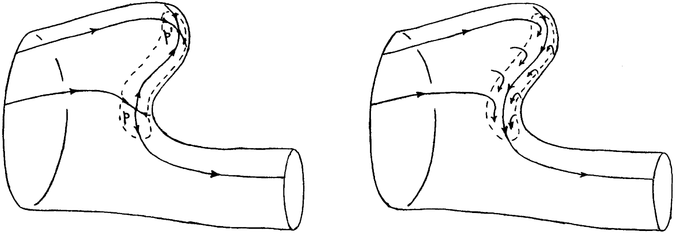

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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homology) and quantifies their *persistence*

Combining persistence and Morse theory

For a Morse function:

- critical points correspond to endpoints of barcode intervals

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Proposition

The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
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- *satisfying $\|f_\delta - f\|_\infty \leq \delta$ and*
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Does not generalize to higher-dimensional manifolds!

Functional topology

When was persistent homology discovered?

- H. Edelsbrunner, D. Letscher, and A. Zomorodian
  Topological persistence and simplification
  Foundations of Computer Science, 2000
- V. Robbins
  Computational Topology at Multiple Resolutions.
  PhD thesis, University of Colorado Boulder, 2000
- P. Frosini
  A distance for similarity classes of submanifolds of a Euclidean space
  Bulletin of the Australian Mathematical Society, 1990.
- S. A. Barannikov.
  The framed Morse complex and its invariants.
  In *Singularities and bifurcations, Adv. Soviet Math. (vol. 21)*, 1994.

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Annals of Mathematics
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

By Marston Morse

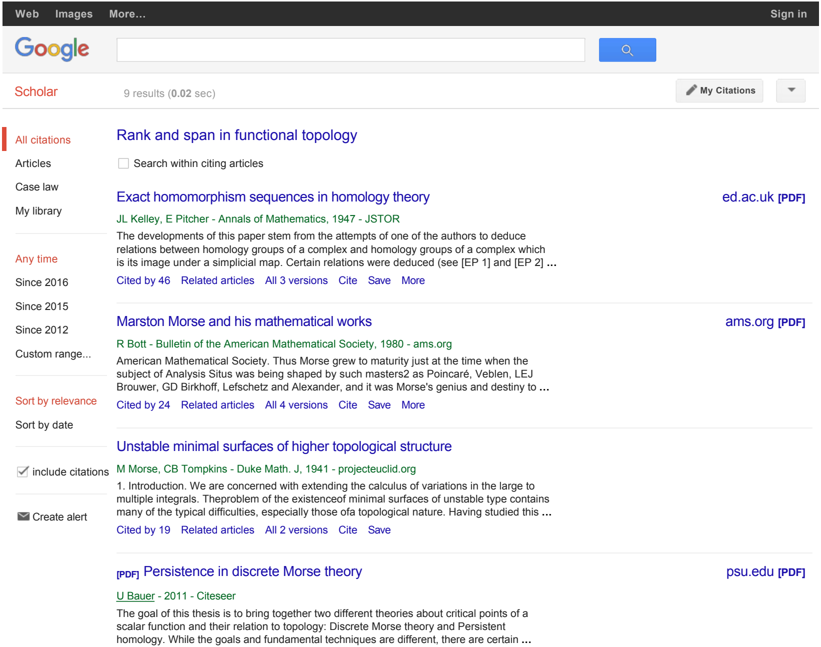
(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are

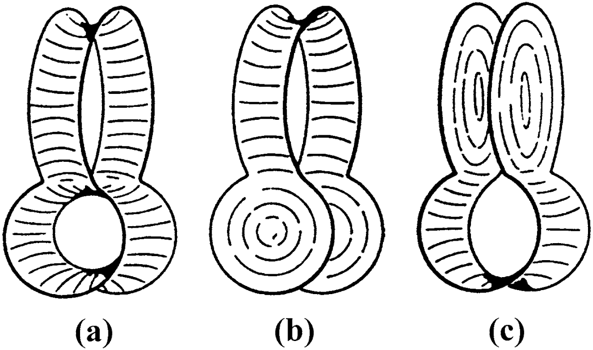
When was persistent homology discovered first?



Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find an immersed disk of least area spanned by a given closed Jordan curve.



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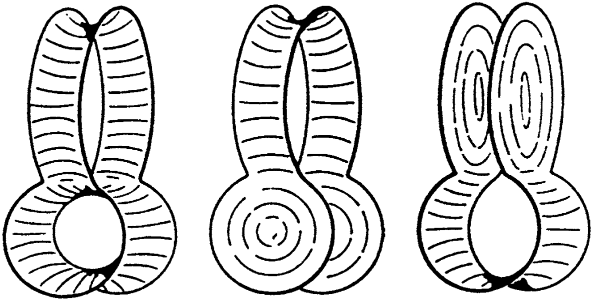

(a) (b) (c)

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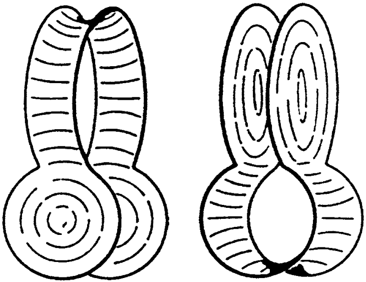
Solution by Douglas (1930):

- identifies minimal surfaces with critical points of the *Douglas functional*

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.



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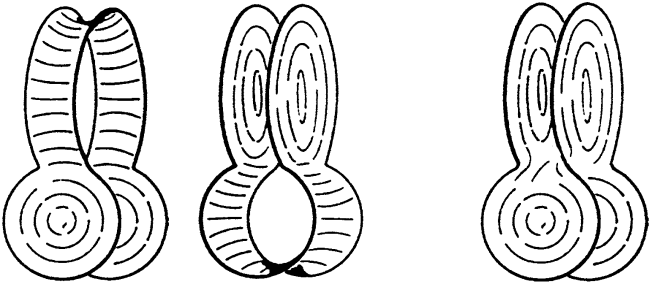
Assume that a given curve bounds two separate stable minimal surfaces. Then there also exists an unstable *minimal surface bounding that curve (a critical point that is not a local minimum).*



Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if for every $s < t$ the structure map $M_s \to M_t$ has finite rank.

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- A q-tame persistence module
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- Morse's goal, in modern language:
  - Sufficient conditions for sublevel sets to have q-tame persistence,
  - which are satisfied by the minimal surface functional

Q-tameness from local connectivity

Theorem (Morse, 1937)

If a function $f\colon X \to \mathbb{R}$ on a metric space X is bounded below and the sublevel set filtration is compact and weakly locally connected, *then it has q-tame persistent Vietoris homology.*

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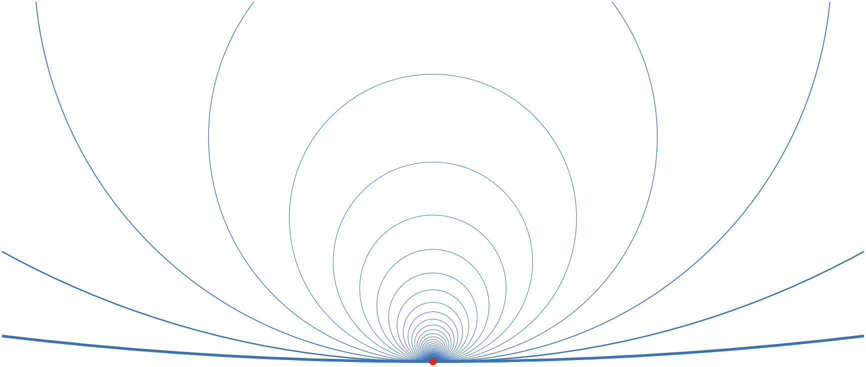

Homologically locally small filtrations

Definition

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is *homologically locally connected (HLC)* if

- for any point $x \in X$, any values $f(x) < s < t$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$,

there is

- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \hookrightarrow V$ induces a zero map on homology.

A sufficient condition for q-tame persistence

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel sets of a function $f: X \to \mathbb{R}$ *are compact and HLC, then their persistent homology is q-tame.*

- f is not required to be continuous

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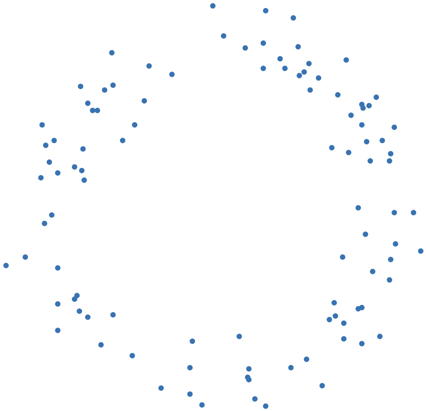
- Conditions are satisfied by the Douglas functional
- Fixes the gap in Morse/Tompkins' proof

Persistence of Vietoris–Rips complexes

Vietoris–Rips complexes



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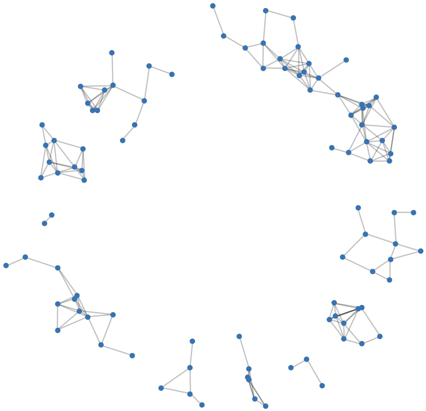

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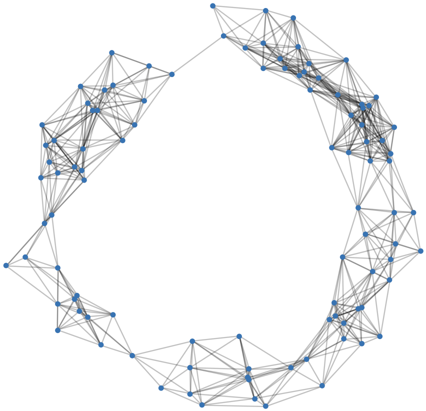

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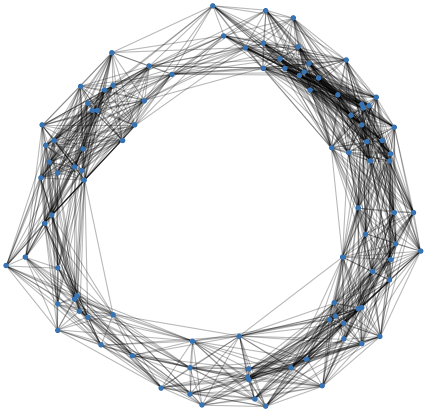

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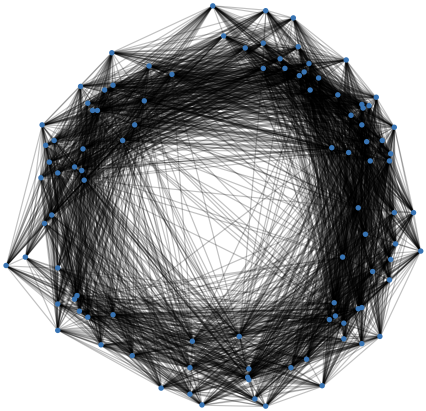

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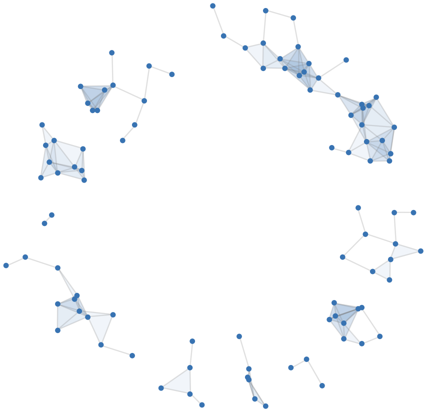

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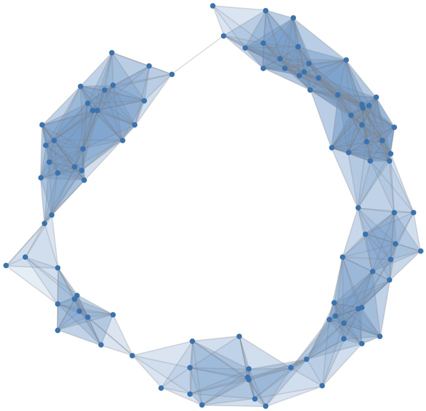

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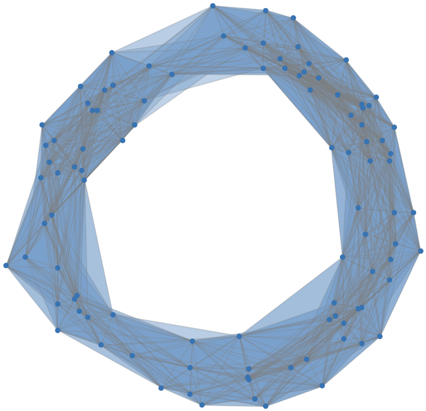

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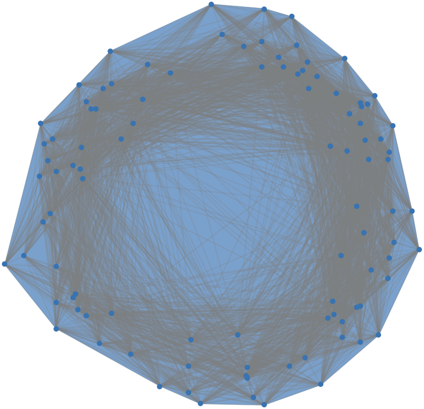

Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at scale $t > 0$ is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \emptyset \text{ finite}, \operatorname{diam} S \leq t\}.$$

An example computation

Example data set:

- 192 points on $\mathbb{S}^2$
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- Dionysus (Duke): 615 seconds, 3.4 GB
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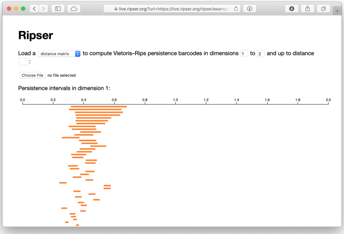
- try live.ripser.org



Apparent pairs:

Ripser uses the following construction for a computational shortcut:

Definition

In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an *apparent pair* if

- σ is the latest proper face of τ, and
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Proposition

The apparent pairs form a discrete gradient.

Proposition

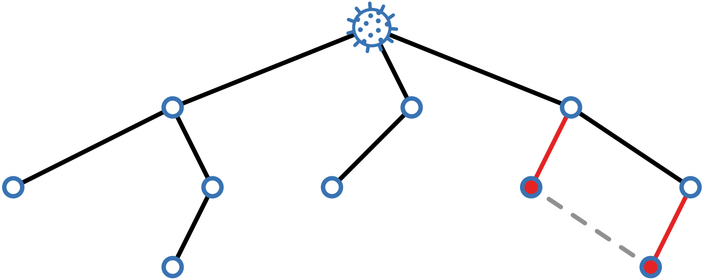
The apparent pairs form persistence pairs: the cycle $\partial\tau$ gives rise to an interval $[f(\sigma), f(\tau))$ in the persistence barcode.

The diameter-lexicographic filtration

We use the *lexicographic refinement* of the Vietoris–Rips filtration:

- Choose a total order on the vertices
- Order simplices by diameter
- Order simplices with same diameter by lexicographic vertex order

Topology of viral evolution



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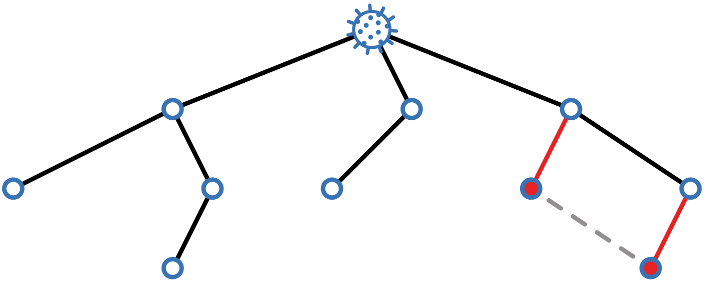

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Observation: Ripser runs unusually fast on genetic distance data

- SARS-CoV2 spike protein: 25556 data points (2.8×10^{12} simplices for H_1)

Can we make it run even faster?

The Rips Contractibility Lemma

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

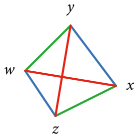
- What about non-geodesic spaces?
- In particular, finite metric spaces?
- Connection to Ripser?

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



- The hyperbolic plane is $\ln 2$-hyperbolic

- 0-hyperbolic spaces are subspaces of trees

Rips contractibility for non-geodesic spaces

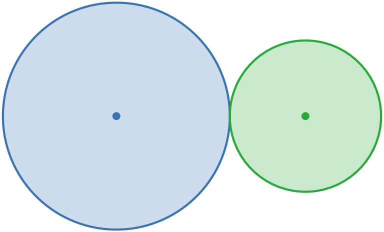
Theorem (B, Roll 2021)

Let X be a finite δ-hyperbolic space. Then there is a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

Geodesic defect



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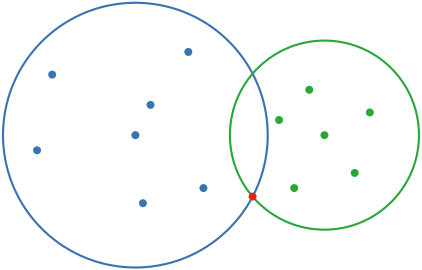

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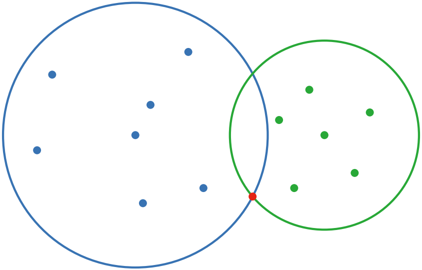

Definition (Bonk, Schramm 2000)

A metric space X is *ν-geodesic* if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{r+\nu}(y) \neq \varnothing.$$

We call the infimum of all such ν the *geodesic defect* of X.

Bounds on the geodesic defect

ν: geodesic defect of finite (X, d)

- lower bound:

$$\nu \geq \frac{1}{2} \min_{x \neq y \in X} d(x, y)$$

- upper bound:

$$\nu \leq d_H(X, Z)$$

for $X \subseteq Z$ an ambient geodesic space

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Possible ambient spaces for X:

- $\ell^\infty(X)$ (functions $X \to \mathbb{R}$)
    - *Kuratowski embedding* $e: X \to \ell^\infty(X)$, $x \mapsto (d_x = d(x, -): X \to \mathbb{R}$
- injective hull (tight span): subspace of $\ell^\infty(X)$

The tight span of a metric space

Given a metric space (X, d), the *injective hull* is

$$E(X) = \{f\colon X \to \mathbb{R} \mid f(x) + f(y) \geq d(x,y), f(x) = \sup_{y \in X}(d(x,y) - f(y))\},$$

equipped with the metric induced by the sup-norm.

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Properties:

- geodesic
- hyperconvex (metric balls have the Helly property)
- injective (in the category of metric spaces) [Aronszain–Panichpakdi 1956]
- contains the Kuratowski embedding $e(X)\colon x \mapsto d_x$
- minimal space with above properties
- contractible [Isbell 1962]

Density of Kuratowski embedding

Theorem (Lang 2013 + ϵ)

Let X be a δ-hyperbolic ν-geodesic metric space. Then the injective hull $E(X)$ is δ-hyperbolic, and every point in $E(X)$ has distance at most $2\delta + \nu$ to $e(X)$.

Corollary (Lim, Mémoli, Okutan 2020 + ϵ)

$\mathrm{Rips}_t(X)$ *is contractible for all* $t > 4\delta + 2\nu$.

Proof outline.

- $\mathrm{Rips}_t(X) = \mathrm{Cech}_r(e(X), E(X))$ (hyperconvexity)
- For $r > 2\delta + \nu$: $(B_r(x))_{x \in e(X)}$ is a good cover of $E(X)$ (density)
- $\mathrm{Cech}_r(e(X), E(X)) \simeq E(X)$ (nerve theorem)
- $E(X) \simeq \{*\}$ (contractibility) □

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- The discrete gradient has the non-critical intervals $[e, \Delta_e]$, where
  - $e \in \binom{V}{2} \setminus E$ is any non-tree edge
  - Δ_e is the unique maximal cofaces with $\operatorname{diam} \Delta_e = l(e)$
- Only vertices V and edges E are critical

Corollary (B, Roll 2021)

The discrete gradient induces collapses

$$\mathrm{Rips}_t(X) \searrow T_t \qquad \textit{for all } t \in \mathbb{R}, \textit{ with}$$

$$\mathrm{Rips}_t(X) \searrow T \searrow \{*\} \qquad \textit{for all } t \geq \max l(E), \textit{ and}$$

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \qquad \textit{for all } (t, u] \cap l(E) = \varnothing.$$

In particular, the persistent homology is trivial in degrees > 0.

Arbitrary tree metrics

Example: phylogenetic trees

- non-generic tree metric
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There is still a compatible gradient:

- independent of choices (*canonical gradient*)
- for generic trees: equals the diam gradient
- only V, E critical
- induces the same collapses

Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
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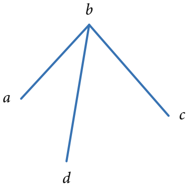

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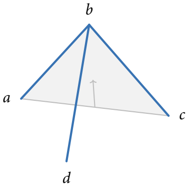

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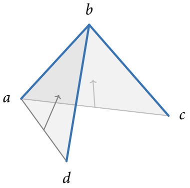

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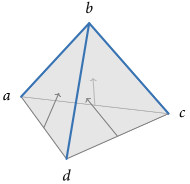

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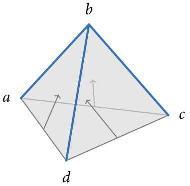

- The canonical gradient (from the previous slide) refines the perturbed gradient (for any choice of total order)
- Hence, the perturbed gradient induces the same collapses

Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric spaces for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

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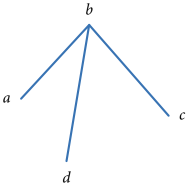
$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow T_t$$

for all $u > t$ such that no tree edge $e \in E$ has length $l(e) \in (t, u]$.



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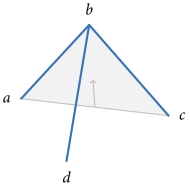

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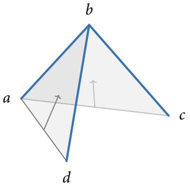

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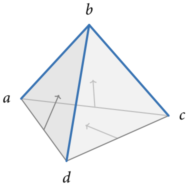

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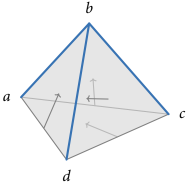

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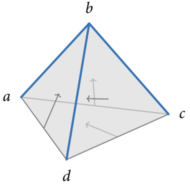

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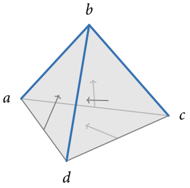

- Explains why Ripser is very fast on genetic distances (tree-like)

U. Bauer, M. Kerber, F. Roll, and A. Rolle

A Unified View on the Functorial Nerve Theorem and its Variations

Preprint, arXiv:2203.03571, 2022

R. Forman

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Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

Preprint, arXiv:2112.06781, 2022

U. Bauer, A. M. Medina-Mardones, and M. Schmahl

Persistence in functional topology and a correction to a theorem of Morse

Preprint, arXiv:2107.14247, 2021

D. Cohen-Steiner, H. Edelsbrunner, and J. Harer.

Stability of persistence diagrams

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Ripser: efficient computation of Vietoris–Rips persistence barcodes

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